

TENSOR FACTORIZATION AND SPIN CONSTRUCTION FOR KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper we discuss the “Factorization phenomenon” which occurs when a representation of a Lie algebra is restricted to a subalgebra, and the result factors into a tensor product of smaller representations of the subalgebra. We analyze this phenomenon for symmetrizable Kac-Moody algebras (including finite-dimensional, semi-simple Lie algebras). We present a few factorization results for a general embedding of a symmetrizable Kac-Moody algebra into another and provide an algebraic explanation for such a phenomenon using Spin construction. We also give some application of these results for semi-simple finite dimensional Lie algebras.

We extend the notion of Spin functor from finite-dimensional to symmetrizable Kac-Moody algebras, which requires a very delicate treatment. We introduce a certain category of orthogonal \mathfrak{g} -representations for which, surprisingly, the Spin functor gives a \mathfrak{g} -representation in Bernstein-Gelfand-Gelfand category \mathcal{O} . Also, for an integrable representation Spin produces an integrable representation. We give the formula for the character of Spin representation for the above category and work out the factorization results for an embedding of a finite dimensional semi-simple Lie algebra into its untwisted affine Lie algebra. Finally, we discuss classification of those representations for which Spin is irreducible.

INTRODUCTION

The *factorization phenomenon* occurs when a representation of a Lie algebra $\tilde{\mathfrak{g}}$ is restricted to a subalgebra \mathfrak{g} , and the result factors into a tensor product of \mathfrak{g} -representations:

$$V \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong V_1 \otimes V_2 \otimes \cdots .$$

We will consider general embeddings of symmetrizable Kac-Moody algebras $\mathfrak{g} \subset \tilde{\mathfrak{g}}$.

This phenomenon has been widely studied when $\tilde{\mathfrak{g}}$ is an affine Lie algebra and \mathfrak{g} its underlying finite-dimensional subalgebra; see [FL1],[FL2],[KMN],[HK],[Ka] and [OSS]. In this case, Fourier and Littelmann [FL1] have shown that every irreducible $\tilde{\mathfrak{g}}$ -representation factors into a tensor product of infinitely many \mathfrak{g} -representations. Their proof by character computations is essentially combinatorial. Our work aims toward an algebraic framework in which factorization appears functorially and in a more general context, treating finite and infinite dimensional Lie algebras simultaneously.

We define a large class of representations which exhibit tensor factorization. First we give some motivation for this class in terms of the characters of its representations. For now, we consider embedding of one semi-simple finite dimensional Lie algebra into another, but we will see later that the arguments also work for embeddings of symmetrizable Kac-Moody algebras with some additional structure. We fix some notation:

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- $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, an embedding of semi-simple finite dimensional Lie algebras.
- $\rho =$ half sum of all positive roots of \mathfrak{g} .
- $\mathcal{W} =$ the Weyl group of \mathfrak{g} .
- $A_\mu = \sum_{w \in \mathcal{W}} \text{sign}(w) e^{w(\mu)}$, the skew symmetrizer of e^μ with respect to \mathcal{W} .
- $R^+ =$ the set of all positive roots of \mathfrak{g} .
- $V(\lambda) =$ irreducible representation with highest weight λ .
- $V \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} =$ restriction of a $\tilde{\mathfrak{g}}$ -representation V to \mathfrak{g} .

Also, for an object a associated to \mathfrak{g} , we use \tilde{a} to denote the corresponding object for $\tilde{\mathfrak{g}}$. For example, $\tilde{\rho}$ denotes the half sum of all positive roots of $\tilde{\mathfrak{g}}$ and $V(\tilde{\rho})$ denotes the irreducible representation of $\tilde{\mathfrak{g}}$ with highest weight $\tilde{\rho}$.

Consider the character of $V(\tilde{\rho})$. By Weyl denominator identity:

$$A_{\tilde{\rho}} = e^{\tilde{\rho}} \prod_{\alpha \in \tilde{R}^+} (1 - e^{-\alpha})$$

and Weyl character formula:

$$\text{Char } V(\lambda) = \frac{A_{\lambda + \tilde{\rho}}}{A_{\tilde{\rho}}},$$

we obtain:

$$\text{Char } V(\tilde{\rho}) = e^{\tilde{\rho}} \prod_{\alpha \in \tilde{R}^+} (1 + e^{-\alpha}).$$

This multiplicative form of the character of $V(\tilde{\rho})$ suggests that the $\tilde{\mathfrak{g}}$ -representation $V(\tilde{\rho})$ when restricted to \mathfrak{g} , might factor into tensor product of \mathfrak{g} -representations.

Now, without loss of generality we may assume that Cartan subalgebra of \mathfrak{g} is contained in Cartan subalgebra of $\tilde{\mathfrak{g}}$ and positive roots of $\tilde{\mathfrak{g}}$ restrict to positive roots of \mathfrak{g} . Then the restriction of the $\tilde{\mathfrak{g}}$ -character $\text{Char } V(\tilde{\rho})$ to \mathfrak{g} will in fact factor as follows:

$$\begin{aligned} \text{Char}(V(\tilde{\rho}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}}) &= \left(e^{\rho} \prod_{\alpha \in R^+} (1 + e^{-\alpha}) \right) \left(e^{(\tilde{\rho} \downarrow - \rho)} \prod_{\alpha \in \tilde{R}^+ \setminus R^+} (1 + e^{-\alpha}) \right) \\ &= \text{Char } V(\rho) \left(e^{(\tilde{\rho} \downarrow - \rho)} \prod_{\alpha \in \tilde{R}^+ \setminus R^+} (1 + e^{-\alpha}) \right) \end{aligned}$$

where \downarrow denotes restriction from $\tilde{\mathfrak{g}}$ to \mathfrak{g} and R^+ is any subset of \tilde{R}^+ which on restriction to \mathfrak{g} forms the set of all positive roots of \mathfrak{g} . Now, if we can find a \mathfrak{g} -representation whose character is the second factor above, we can conclude that for any embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ of semi-simple Lie algebras, the irreducible $\tilde{\mathfrak{g}}$ -representation $V(\tilde{\rho})$, when restricted to \mathfrak{g} , always factors into a tensor product of at least two \mathfrak{g} -representations, one of them being $V(\rho)$.

The \mathfrak{g} -representation whose character is the second factor above is obtained using Panyushev's [P] reduced Spin functor Spin_0 . We will define Spin_0 in §2 (and briefly in §1.3). Basically, Spin for a given Lie-algebra \mathfrak{g} is a functor from the category of all \mathfrak{g} -representations which have a non-degenerate symmetric bilinear form preserved by the action of \mathfrak{g} (called *orthogonal* \mathfrak{g} -representations) to the category of all \mathfrak{g} -representations. By reducing multiplicities in the resulting representation we obtain Spin_0 which has the remarkable property that:

$$\text{Spin}_0(V_1 \oplus V_2) \cong \text{Spin}_0(V_1) \otimes \text{Spin}_0(V_2).$$

It is a well known fact that for any semi-simple finite dimensional Lie algebra \mathfrak{g} , the representation $V(\rho)$ can be realized as Spin_0 of adjoint representation of \mathfrak{g}

(which is indeed orthogonal due to the invariant Killing form). Thus, for the Lie algebra $\tilde{\mathfrak{g}}$,

$$V(\tilde{\rho}) = \text{Spin}_0(\tilde{\mathfrak{g}}).$$

When we restrict to \mathfrak{g} , it turns out that Spin_0 commutes with the restriction according to :

$$\text{Spin}_0(\tilde{\mathfrak{g}}) \downarrow \cong 2^r \text{Spin}_0(\tilde{\mathfrak{g}} \downarrow),$$

where r is the number of positive roots of $\tilde{\mathfrak{g}}$ which restrict to zero. Now, $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{g}^\perp$ as \mathfrak{g} -representation, where \perp denotes the orthogonal complement with respect to the Killing form. Therefore,

$$\begin{aligned} V(\tilde{\rho}) \downarrow_{\mathfrak{g}} &\cong 2^r \text{Spin}_0(\mathfrak{g} \oplus \mathfrak{g}^\perp) \\ &\cong 2^r [\text{Spin}_0(\mathfrak{g}) \otimes \text{Spin}_0(\mathfrak{g}^\perp)], \end{aligned}$$

by the property of Spin_0 mentioned above. So,

$$\begin{aligned} V(\tilde{\rho}) \downarrow_{\mathfrak{g}} &\cong \text{Spin}_0(\mathfrak{g}) \otimes [2^r \text{Spin}_0(\mathfrak{g}^\perp)] \\ &\cong V(\rho) \otimes [2^r \text{Spin}_0(\mathfrak{g}^\perp)], \end{aligned}$$

as $V(\rho)$ is isomorphic to $\text{Spin}_0(\mathfrak{g})$. Hence, we get the tensor factorization of the restricted $V(\tilde{\rho})$. This is the content of Theorem 1 in §1.2 where it is extended to embedding of symmetrizable Kac-Moody algebras with some additional structure. The detailed proof is given later.

From this, using Weyl character formula, we can obtain a tensor factorization of the $\tilde{\mathfrak{g}}$ -representation $V(2\tilde{\mu} + \tilde{\rho})$ for *any* dominant weight $\tilde{\mu}$, which forms the content of Theorem 2 in §1.2. In §1.3, we state some important properties of the Spin functor. We describe some consequences of the above theorems for finite dimensional semi-simple Lie algebras and untwisted affine Lie algebras in §1.4 and §1.5 respectively. In §1.6, for a subclass of orthogonal representations of untwisted affine Lie algebras, called affinized representations, we classify those whose Spin_0 is irreducible. Following Panyushev [P] these are called coprimary representations.

1. MAIN RESULTS

1.1. Background for symmetrizable Kac-Moody algebras. An $n \times n$ matrix, $A = (a_{ij})$, is called a *generalized Cartan matrix* if:

- (1) $a_{ii} = 2$ for all $i = 1, 2, \dots, n$.
- (2) a_{ij} is a non-positive integer for all $i \neq j$.
- (3) $a_{ij} = 0$ implies $a_{ji} = 0$ for all $i \neq j$.

For any $n \times n$ matrix $A = (a_{ij})$ of rank l , we define a *realization* of A as a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where \mathfrak{h} is a complex vector space, $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets in \mathfrak{h}^* and \mathfrak{h} respectively, satisfying the following three conditions:

- (1) Both sets Π and Π^\vee are linearly independent.
- (2) $\alpha_j(\alpha_i^\vee) = a_{ij}$ for all $i, j = 1, 2, \dots, n$.
- (3) $\dim(\mathfrak{h}) = 2n - l$.

Two realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ are called isomorphic if there exists a vector space isomorphism $\phi : \mathfrak{h} \rightarrow \mathfrak{h}_1$ such that $\phi(\Pi^\vee) = \Pi_1^\vee$ and $\phi^*(\Pi_1) = \Pi$. There exists a unique (up to isomorphism) realization of every $n \times n$ matrix. The realizations of two matrices A and B are isomorphic if B can be obtained from A by a permutation of the indexing set [Ka, Proposition 1.1].

An $n \times n$ matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ and a symmetric matrix B such that $A = DB$.

Definition. (Symmetrizable Kac-Moody algebra) Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A . A symmetrizable Kac-Moody algebra, \mathfrak{g} , associated to A is defined as the Lie algebra on generators $X_{\pm i}$ ($i = 1, \dots, n$), all $H \in \mathfrak{h}$ with the following defining relations:

- (1) $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$.
- (2) $[H, X_{\pm i}] = \pm \alpha_i(H) X_{\pm i}$ for all $i = 1, \dots, n$ and $H \in \mathfrak{h}$.
- (3) $[X_i, X_{-j}] = \delta_{ij} \alpha_i^\vee$ for all $i, j = 1, \dots, n$.
- (4) $\text{ad}(X_{\pm i})^{1-a_{ij}}(X_{\pm j}) = 0$ for all $i, j = 1, \dots, n$.

Here, $\text{ad}(X)(\cdot) := [X, \cdot]$ and \mathfrak{h} is called the Cartan subalgebra of \mathfrak{g} .

Let \mathfrak{g} be symmetrizable Kac-Moody algebra. We define a non-degenerate symmetric bilinear form (\cdot, \cdot) on \mathfrak{h} which can be extended (See [Ka, Thm 2.2]) to a non-degenerate symmetric bilinear form on whole of \mathfrak{g} such that (\cdot, \cdot) is preserved by the adjoint action of \mathfrak{g} , that is:

$$([X, Y], Z) + (Y, [X, Z]) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Let A be a symmetrizable generalized Cartan matrix with a fixed decomposition $A = DB$ (See the definition of a symmetrizable matrix above). Let $\mathfrak{h}' := \bigoplus_{i=1}^n \mathbb{C} \alpha_i^\vee$. Fix a complementary space \mathfrak{h}'' to \mathfrak{h}' in \mathfrak{h} and define:

$$\begin{aligned} (\alpha_i^\vee, H) &= \alpha_i(H) \epsilon_i \quad \forall H \in \mathfrak{h}; \\ (H_1, H_2) &= 0 \quad \forall H_1, H_2 \in \mathfrak{h}''. \end{aligned}$$

1.2. Factorization Theorems. We now state our main factorization results using Spin construction. We consider embeddings, $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, of symmetrizable Kac-Moody algebras. Our analysis deals with finite as well as infinite dimensional representations. For example, we consider infinite dimensional irreducible representations of an affine Lie algebra $\tilde{\mathfrak{g}}$ with finite dimensional weight spaces. If we restrict such a representation to a finite dimensional Lie algebra \mathfrak{g} , the \mathfrak{g} -weight spaces no longer remain finite dimensional. To avoid this, we enlarge the Cartan subalgebra of \mathfrak{g} by one dimension by augmenting an element d from Cartan subalgebra of $\tilde{\mathfrak{g}}$ so that for certain class of $\tilde{\mathfrak{g}}$ -representations, the restricted representation to $\mathfrak{g} \oplus \mathbb{C}d$ has finite-dimensional weight spaces. This gives rise to the following notions: an *augmented symmetrizable Kac-Moody algebra*, a *d-embedding* of such algebras, say $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, and *d-finite* representations so that any *d-finite* $\tilde{\mathfrak{g}}$ -representation, when restricted to \mathfrak{g} , has finite dimensional weight spaces.

Definition. (Augmented symmetrizable Kac-Moody algebra \mathfrak{g}) A Lie algebra \mathfrak{g} is called an augmented symmetrizable Kac-Moody algebra if \mathfrak{g} has a certain distinguished element d in the Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that either:

- \mathfrak{g} itself is a symmetrizable Kac-Moody algebra and $\alpha_i(d) \in \mathbb{Z}_{>0}$ for all $\alpha_i \in \Pi$,
- or $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{C}d$ where:
 - \mathfrak{g}_1 is a symmetrizable Kac-Moody algebra.
 - for each root vector $X_{\pm\alpha}$ of \mathfrak{g}_1 , $[d, X_{\pm\alpha}] = \pm c_\alpha X_{\pm\alpha}$, for some *positive* integer c_α .

Remark. When $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{C}d$, we can extend the action of the root α of \mathfrak{g}_1 to $(\mathfrak{h}_1 \oplus \mathbb{C}d)$ by defining $\alpha(d) := c_\alpha$. Thus $\mathfrak{h} := \mathfrak{h}_1 \oplus \mathbb{C}d$ is the Cartan subalgebra of \mathfrak{g} .

Example: Let $\mathfrak{g}_1 \cong \mathfrak{sl}_2 \mathbb{C} := \mathbb{C}H_\alpha \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$ with usual bracket relations. Define $[d, H_\alpha] := 0$ and $[d, X_{\pm\alpha}] := \pm X_{\pm\alpha}$, so that $\alpha(d) := 1$. Then, $\mathfrak{g} := \mathfrak{g}_1 \oplus \mathbb{C}d$ is an augmented symmetrizable Lie algebra.

Definition. (d -embedding) An embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ of augmented symmetrizable Kac-Moody algebras with distinguished element d and \tilde{d} and Cartan subalgebras \mathfrak{h} and $\tilde{\mathfrak{h}}$ respectively, is called a d -embedding if :

- $d = \tilde{d}$,
- $\mathfrak{h} \subset \tilde{\mathfrak{h}}$ and
- positive roots of \mathfrak{g} are restrictions of positive roots of $\tilde{\mathfrak{g}}$.

Let \mathfrak{g} be an augmented symmetrizable Kac-Moody algebra with distinguished element d in the Cartan subalgebra \mathfrak{h} and weight lattice \mathcal{P} . For a \mathfrak{g} -representation V and $\Lambda \in \mathcal{P}$, let $V^{(\Lambda)} := \{v \in V : H(v) = \Lambda(H)v \ \forall \ H \in \mathfrak{h}\}$ denote the corresponding weight space of V . Λ is called a *weight of V* if $V^{(\Lambda)} \neq \{0\}$.

For the distinguished element d in the Cartan subalgebra of \mathfrak{g} , we say a \mathfrak{g} -representation V is *d -finite* if :

- $\Lambda(d) \in \mathbb{Z} \setminus \{0\}$ for all non-zero weights Λ of V and
- $\bigoplus_{\Lambda(d)=k} V^{(\Lambda)}$ is finite-dimensional for each $k \in \mathbb{Z}$.

As we mentioned in the Introduction, the input for the Spin functor is an orthogonal representation which we defined for semi-simple finite dimensional Lie algebras. The same definition extends to the augmented symmetrizable Kac-Moody algebra too. For an augmented symmetrizable Kac-Moody algebra \mathfrak{g} , a \mathfrak{g} -representation V is called *orthogonal* if there exists a non-degenerate symmetric bilinear form Q on V , invariant under the action of \mathfrak{g} , that is, $Q(Xu, v) + Q(u, Xv) = 0$ for all $u, v \in V$ and $X \in \mathfrak{g}$. For example, the action of a symmetrizable Kac-Moody algebra on itself by brackets, called adjoint representation, is orthogonal due to the invariant bilinear form (see §1.1).

Next we define the adjoint representation of an augmented symmetrizable Kac-Moody algebra in such a way that it is orthogonal, so that we can apply the Spin functor to it (see the Introduction). Adjoint representation for an augmented symmetrizable Kac-Moody algebra \mathfrak{g} with distinguished element d is already defined if \mathfrak{g} is itself a symmetrizable Kac-Moody algebra. So, let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{C}d$. In this case, the action of \mathfrak{g} on \mathfrak{g}_1 by brackets is defined as the *adjoint representation* of \mathfrak{g} . We can show that the action of the distinguished element d preserves the bilinear form on \mathfrak{g}_1 and thus \mathfrak{g}_1 is orthogonal as a \mathfrak{g} -representation.

Remark. It is easy to show that for a d -embedding, $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, the adjoint representation of $\tilde{\mathfrak{g}}$ is d -finite and orthogonal both as a $\tilde{\mathfrak{g}}$ -representation and a \mathfrak{g} -representation.

Theorems 1 and 2 (given below) describe a class of representations which exhibit the factorization phenomenon (see the Introduction). We will follow the notations used in the Introduction except that $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ will denote a d -embedding of two augmented symmetrizable Kac-Moody algebras and ρ and $\tilde{\rho}$ will denote the sum of all fundamental weights of \mathfrak{g} and $\tilde{\mathfrak{g}}$ respectively.

Theorem 1. *For a d -embedding, $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, of augmented symmetrizable Kac-Moody algebras, suppose the adjoint representation of $\tilde{\mathfrak{g}}$ decomposes into orthogonal \mathfrak{g} -representations as: $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \dots$. Then the $\tilde{\mathfrak{g}}$ -representation $V(\tilde{\rho})$, when restricted from $\tilde{\mathfrak{g}}$ to \mathfrak{g} , factors into a tensor product of \mathfrak{g} -representations as:*

$$V(\tilde{\rho}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes W_1 \otimes W_2 \otimes \dots$$

with $W_j = \text{Spin}_0(\mathfrak{p}_j)$, where Spin_0 is the reduced Spin functor defined in §2.

In the finite-dimensional case, the Theorem is closely related to the results of Kostant [K1, K2].

Theorem 1 leads to a large class of representations exhibiting tensor factorization.

Theorem 2. *Let $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ be as in Theorem 1, and let $\tilde{\mu}$ be a dominant weight of $\tilde{\mathfrak{g}}$. Then the $\tilde{\mathfrak{g}}$ representation $V(2\tilde{\mu} + \tilde{\rho})$, when restricted to \mathfrak{g} , factors into a tensor product of \mathfrak{g} -representations which include the same W_j as in Theorem 1. The other factor can be expressed in terms of the irreducible decomposition of the restricted $V(\tilde{\mu})$.*

That is, if we let:

$$V(\tilde{\mu}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong \bigoplus_i V(\mu_i),$$

then:

$$V(2\tilde{\mu} + \tilde{\rho}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong \left(\bigoplus_i V(2\mu_i + \rho) \right) \otimes W_1 \otimes W_2 \otimes \cdots.$$

We will prove these theorems in §3.

1.3. Basic properties of the Spin functor. We now describe the basic properties of Spin functor, reserving the more technical discussion for §2. In the finite-dimensional case the construction is quite simple, and was examined by Panyushev [P]. For an n -dimensional vector space V with a non-degenerate symmetric bilinear form, recall that the orthogonal Lie algebra $\mathfrak{so}(V)$ has a representation on the $2^{\lfloor n/2 \rfloor}$ -dimensional space $\text{Spin}(V) := \wedge^{\bullet} V^+$, the total wedge space or exterior algebra of a maximal isotropic subspace $V^+ \subset V$.

Let \mathfrak{g} be a semi-simple, finite dimensional Lie algebra. For an orthogonal \mathfrak{g} -representation V , \mathfrak{g} acts by orthogonal matrices: that is, through an embedding $\mathfrak{g} \subset \mathfrak{so}(V)$. Restricting the action of $\mathfrak{so}(V)$ makes $\text{Spin}(V)$ a representation of \mathfrak{g} . If the zero weight space of V has dimension r , it turns out that $\text{Spin}(V)$ can be decomposed as the direct sum of $2^{\lfloor r/2 \rfloor}$ copies of a smaller representation, which we call $\text{Spin}_0(V)$.

Now let \mathfrak{g} be an augmented symmetrizable Kac-Moody algebra. In §2, we will define the \mathfrak{g} -representation $\text{Spin}(V) = \wedge^{\bullet} V^+$ for V in the category of all d -finite and orthogonal (possibly infinite-dimensional) \mathfrak{g} -representations. We will prove that the output, $\text{Spin}(V)$, will be a d -finite \mathfrak{g} -representation in the category $\mathcal{O}_{\text{weak}}$ (defined below). Category $\mathcal{O}_{\text{weak}}$ contains the Bernstein-Gelfand-Gelfand category \mathcal{O} and has similar properties. Further if the input representation V is root finite (defined below) then we will prove, $\text{Spin}(V)$ belongs to \mathcal{O} . If the zero weight space of V is even, $\text{Spin}(V)$ decomposes into direct sum of \mathfrak{g} -representation which we call *half-Spin representations* $\wedge^{\text{even}} V^+$ and $\wedge^{\text{odd}} V^+$. We also denote these by $\text{Spin}^{\text{even}}(V)$ and $\text{Spin}^{\text{odd}}(V)$.

We define a partial ordering \leq called root order, on the the weight lattice \mathcal{P} of \mathfrak{g} as follows: We say, $\beta \leq \gamma$ in the **root order** if $\gamma - \beta = \sum_{\alpha} c_{\alpha} \alpha$ where α is a simple positive root of \mathfrak{g} and $c_{\alpha} \in \mathbb{Z}_{\geq 0}$ for all α .

For an augmented symmetrizable Kac-Moody algebra \mathfrak{g} and a \mathfrak{g} -representation V define:

$$M_V := \text{Set of all weights of } V \text{ maximal in the root order.}$$

Definition. (Category $\mathcal{O}_{\text{weak}}$ of \mathfrak{g} -representations) $\mathcal{O}_{\text{weak}}$ consists of all \mathfrak{g} -representations V such that for each weight β of V :

- (1) the weight space $V^{(\beta)}$ is finite dimensional and
- (2) there exists $\lambda \in M_V$ such that $\beta \leq \lambda$.

Remark. The morphisms in $\mathcal{O}_{\text{weak}}$ are \mathfrak{g} -representations homomorphisms. Following fact can be deduced, using [K, Thm 10.7], that for a representation V in

$\mathcal{O}_{\text{weak}}$ which is integrable (meaning simple root vectors act locally nilpotently), the isomorphism class of V is determined by its character.

The well-known *Bernstein-Gelfand-Gelfand category* \mathcal{O} of \mathfrak{g} -representations can be defined as a subcategory of $\mathcal{O}_{\text{weak}}$:

$$\mathcal{O} := \{V \in \mathcal{O}_{\text{weak}} : M_V \text{ is a finite set}\}.$$

It is worth noting that \mathcal{O} can be defined to consist of \mathfrak{g} -representations V such that V has finite-dimensional weight spaces and there exists a finite set F , a subset of weight lattice \mathcal{P} of \mathfrak{g} , so that for each weight β of V we can find $\lambda \in F$ with $\beta \leq \lambda$.

Let $\{\alpha_i\}_{i=1}^n$ denote the simple positive roots of an augmented symmetrizable Kac-Moody algebra \mathfrak{g} with distinguished element d . Let \mathfrak{h}^* be the dual Cartan subalgebra of \mathfrak{g} . Define the root cone:

$$C := \left\{ \sum_{i=1}^n a_i \alpha_i \in \mathfrak{h}^* : a_i \in \mathbb{R}_{\geq 0} \ \forall i \text{ or } a_i \in \mathbb{R}_{\leq 0} \ \forall i \right\}.$$

Definition. Root finite \mathfrak{g} -representation We say a \mathfrak{g} -representation V is root-finite if for every weight Λ of V , $V^{(\Lambda)}$ is finite dimensional, $\Lambda(d) \in \mathbb{Z} \setminus \{0\}$ for $\Lambda \neq 0$ and there are only finitely many weights of V in $\mathfrak{h}^* \setminus C$.

Propositions 1 and 2 give some basic properties of $\text{Spin}(V)$.

Proposition 1. *Let \mathfrak{g} be an augmented symmetrizable Kac-Moody algebra with distinguished element d . Let V , V_1 and V_2 be d -finite and orthogonal \mathfrak{g} -representations.*

- (1) $\text{Spin}(V)$ is d -finite and belongs to $\mathcal{O}_{\text{weak}}$.
- (2) V is integrable $\Rightarrow \text{Spin}(V)$ is integrable and $\text{Spin}(V) \cong W^{\oplus r}$ for some \mathfrak{g} -representation W called $\text{Spin}_0(V)$. Here $r = \lfloor m_0/2 \rfloor$ where m_0 is the dimension of the zero weight space, $V^{(0)}$ of V .
- (3) Let $m_i := \dim(V_i^{(0)})$ for $i = 1, 2$.
If at least one of m_1 or m_2 is even, then:

$$\text{Spin}(V_1 \oplus V_2) \cong \text{Spin}(V_1) \otimes \text{Spin}(V_2).$$

If both m_1 and m_2 are odd, then:

$$\text{Spin}^{\text{even}}(V_1 \oplus V_2) \cong \text{Spin}(V_1) \otimes \text{Spin}(V_2) \cong \text{Spin}^{\text{odd}}(V_1 \oplus V_2).$$

and

$$\text{Spin}(V_1 \oplus V_2) \cong (\text{Spin}(V_1) \otimes \text{Spin}(V_2))^{\oplus 2}.$$

- (4) If V_1 and V_2 are integrable then

$$\text{Spin}_0(V_1 \oplus V_2) \cong \text{Spin}_0(V_1) \otimes \text{Spin}_0(V_2).$$

- (5) W is root-finite $\Rightarrow W$ is d -finite.
- (6) V is root-finite $\Leftrightarrow \text{Spin}(V) \in \mathcal{O}$
- (7) For adjoint representation \mathfrak{g} ,

$$\text{Spin}_0(\mathfrak{g}) \cong V(\rho).$$

Let

\mathcal{I}_O = Category of all d -finite and orthogonal \mathfrak{g} -representations ,

\mathcal{I}_R = Category of all root-finite and orthogonal \mathfrak{g} -representations.

Then, by Proposition 1(5),

$$\mathcal{I}_R \subset \mathcal{I}_O$$

and by Proposition 1(1) and 1(6), $\text{Spin}(V)$ is a functor from the category \mathcal{I}_O to the category $\mathcal{O}_{\text{weak}}$ and also from category \mathcal{I}_R to category \mathcal{O} . Thus,

$$\begin{array}{ccc} \mathcal{I}_O & \xrightarrow{\text{Spin}} & \mathcal{O}_{\text{weak}} \\ \cup & & \cup \\ \mathcal{I}_R & \xrightarrow{\text{Spin}} & \mathcal{O} \end{array}$$

The following Proposition gives the character of $\text{Spin}_0(V)$ in terms of the character of V .

Proposition 2. *Let V be an integrable \mathfrak{g} -representation in \mathcal{I}_O . Let m_β be the multiplicity of a weight β of V so that the character of V can be written as:*

$$\text{Char } V = \sum_{\beta(d) > 0} m_\beta (e^\beta + e^{-\beta}) + m_0.$$

Then the \mathfrak{g} -representation $\text{Spin}_0(V)$ has the character:

$$\text{Char } \text{Spin}_0(V) = e^\Lambda \prod_{\beta(d) > 0} (1 + e^{-\beta})^{m_\beta}.$$

Here $\Lambda := \sum_{i=1}^n c_i \Lambda_i$, where $\{\Lambda_i\}_{i=1}^n$ are the fundamental weights and the coefficient c_i is defined as follows:

$$c_i := \sum \frac{1}{2} m_\beta \beta(\alpha_i^\vee),$$

where the sum is over all weights β of V such that $\beta(d) > 0$ and $s_i(\beta)(d) < 0$. Here s_i denotes the reflection in the plane perpendicular to the simple root α_i . Because of the d -finiteness of V , c_i has finitely many nonzero terms.

Remark. When V is finite dimensional, the Λ simplifies to $\Lambda = \sum_{\beta(d) > 0} \frac{1}{2} m_\beta \beta$.

1.4. Special cases for Finite-dimensional Lie algebras. We give some special cases of Theorems 1 and 2 when $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is an arbitrary embedding of finite-dimensional semi-simple Lie algebras. This embedding can be turned into a d -embedding of augmented symmetrizable Kac-Moody algebras, by appropriately choosing a d in the Cartan subalgebra of \mathfrak{g} .

1.4.1. Principal Specialization. We let $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ be the embedding of a principal three-dimensional subalgebra in the special linear Lie algebra: $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{sl}_n(\mathbb{C})$, defined as $\mathfrak{sl}_2(C) := \mathbb{C}X \oplus \mathbb{C}Y \oplus \mathbb{C}H$, where

$$\begin{aligned} X &:= \sum_{i=1}^{n-1} i E_{i,i+1}, \\ Y &:= \sum_{i=1}^{n-1} (n-i) E_{i+1,i}, \\ H &:= \sum_{i=1}^n (n+1-2i) E_{i,i}. \end{aligned}$$

Here, $E_{i,j}$ denotes the $n \times n$ matrix which has 1 at $(i,j)^{\text{th}}$ place and zero elsewhere.

The character of a \mathfrak{sl}_n -irreducible $V(\mu)$ is the Schur polynomial $S_\mu(x_1, \dots, x_n)$. Its restriction from $\tilde{\mathfrak{g}}$ to \mathfrak{g} corresponds to the *principal specialization* $x_i \downarrow_{\tilde{\mathfrak{g}}}^{\mathfrak{g}} = q^{i-1}$, where $q = e^\alpha$ for α the simple root of \mathfrak{g} . Theorem 2 implies the following factorization of the specialized Schur function:

Proposition 3.

$$S_{2\mu+\rho}(1, q, q^2, \dots, q^{n-1}) = \left(q^{\binom{n}{3}} (1+q) S_\mu(1, q^2, q^4, \dots, q^{2n-2}) \right) \cdot w_1(q) \cdot w_2(q) \cdots w_{n-2}(q),$$

where $w_k(q) = (1+q)(1+q^2) \cdots (1+q^{k+1})$, $\rho = (n-1, \dots, 1, 0)$ and all $n-1$ factors on the right-hand side of the formula are symmetric unimodal q -polynomials.

Definition. (Symmetric unimodal polynomial) A polynomial of the form, $f(q) = \sum_{i=N}^M a_i q^i$, is symmetric unimodal if $a_{N+i} = a_{M-i}$ for all i , and $a_N \leq \cdots \leq a_K \geq a_{K+1} \geq \cdots \geq a_M$ for some K .

Proposition 3 is a kind of multiplicative analog of a result of Reiner and Stanton which states that for certain pairs λ, μ , the centered difference $S_\lambda(1, \dots, q^{n-1}) - q^N S_\mu(1, \dots, q^{n-1})$ is symmetric unimodal.

1.4.2. Folding of Dynkin diagrams. Let $\tilde{\mathfrak{g}}$ be a simple Lie algebra with Dynkin diagram \tilde{D} . A graph automorphism ϕ of \tilde{D} , induces an automorphism, call it ϕ again, on $\tilde{\mathfrak{g}}$. We let \mathfrak{g} be the fixed subalgebra, under this automorphism, ϕ . Then the Dynkin diagram, D of \mathfrak{g} is called the folding of \tilde{D} . For such an embedding, $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, Theorem 1 implies:

Proposition 4.

$$V(\tilde{\rho}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes [V(e(\rho+\rho_s)+\rho_s) \oplus (a-2)V(0)]^{\otimes a-1},$$

where ρ_s is the half-sum of the positive short roots of \mathfrak{g} , a is the order of the automorphism ϕ , and e is the number of edges $\overset{i}{\bullet} \text{---} \overset{j}{\bullet}$ in \tilde{D} such that ϕ exchanges i and j .

For example, the natural embedding $\mathfrak{so}_{2n+1}\mathbb{C} \subset \mathfrak{sl}_{2n+1}\mathbb{C}$ corresponds to horizontally folding the diagram A_{2n} to obtain B_n :

$$\begin{array}{l} A_{2n} : \quad \overset{1}{\bullet} \text{---} \overset{2}{\bullet} \cdots \overset{n-1}{\bullet} \text{---} \overset{n}{\bullet} \text{---} \overset{n+1}{\bullet} \text{---} \overset{n+2}{\bullet} \cdots \overset{2n-1}{\bullet} \text{---} \overset{2n}{\bullet} \\ B_n : \quad \overset{1}{\bullet} \text{---} \overset{2}{\bullet} \cdots \overset{n-1}{\bullet} \rightrightarrows \overset{n}{\bullet} \end{array}$$

The automorphism is $\phi(i) = 2n-i+1$ of order $a = 2$ with a single folded edge so that $e = 1$. Thus, $V(\tilde{\rho}) \downarrow_{\mathfrak{g}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes V(\rho+2\rho_s)$.

1.5. Factorization Theorems for affine Lie algebras. The most remarkable aspect of our construction appears when $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}}$ is the untwisted affine Lie algebra associated to a finite-dimensional semi-simple algebra \mathfrak{g} :

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Here K is the central element and d the canonical derivation. We also let Λ_0 be the distinguished fundamental weight, and δ the minimal imaginary root. Also, if a is an object associated to \mathfrak{g} , then \hat{a} denotes the corresponding object for $\hat{\mathfrak{g}}$.

Let $\hat{d} := \rho^\vee + hd$ where $h := \sum_{i=0}^n a_i$ is the Coxeter number. Here a_i 's are the numeric labels of the Dynkin diagram of $\hat{\mathfrak{g}}$ in [K, Page 54]) and ρ^\vee is the sum of all fundamental co-weights of \mathfrak{g} , that is $\alpha(\rho^\vee) = 1$ for all simple positive roots α of \mathfrak{g} . Then $\mathfrak{g} \oplus \mathbb{C}\hat{d} \subset \hat{\mathfrak{g}}$ is a \hat{d} -embedding of augmented symmetrizable Kac-Moody algebra.

Let $\hat{V} \in \mathcal{I}_R$ (See §1.3 for definition of \mathcal{I}_R) be an integrable $\hat{\mathfrak{g}}$ -representation of level zero, that is the center K acts by zero. Then, even though the input $\hat{\mathfrak{g}}$ -representation $\hat{V} \in \mathcal{I}_R$ has level zero, by Proposition 2 the output $\text{Spin}(\hat{V})$ is a

representation of *positive* level in the Bernstein-Gelfand-Gelfand category \mathcal{O} . That is, Spin is a functor from the category of graded level zero representations in \mathcal{I}_R , a sub-category of the graded level-zero representations \mathcal{I} examined by Chari and Greenstein [CG], to the positive-level category \mathcal{O} .

Now we introduce a subcategory \mathcal{I}_A of \mathcal{I}_R . We will work out Theorems 1 and 2 for this sub-category.

1.5.1. Affinized representations. For the remainder of this section we will work with $\widehat{\mathfrak{g}}$ -representations in a more restricted class $\mathcal{I}_A \subset \mathcal{I}_R$, the subcategory of affinizations of finite-dimensional orthogonal \mathfrak{g} -representations. That is, for an orthogonal \mathfrak{g} -representation V , its affinization is the $\widehat{\mathfrak{g}}$ -representation

$$\hat{V} := \bigoplus_{k \in \mathbb{Z}} t^k V$$

where the loop algebra acts as $t^l X \cdot t^k v := t^{k+l}(X \cdot v)$ for $X \in \mathfrak{g}$, $v \in V$; the center acts as 0; and the derivation d acts as $t \frac{d}{dt}$. This inherits a non-degenerate symmetric bilinear form from V . Choose a strictly dominant co-weight d_1 in the Cartan subalgebra \mathfrak{h} of \mathfrak{g} such that $\beta(d_1) \in \mathbb{Z} \setminus \{0\}$ for all weights β of V . Since V is finite dimensional, for sufficiently large N , $-N < \beta(d_1), \theta(d_1) < N$ for all weights β of V and highest root θ of \mathfrak{g} . Define $\hat{d} := Nd + d_1$. It can be verified that the weights of \hat{V} are of the form $\Lambda := k\delta + \beta$ for $k \in \mathbb{Z}$ and β a weight of V . Thus for all non-zero weights Λ of \hat{V} , $\Lambda(\hat{d}) \in \mathbb{Z} \setminus \{0\}$. Now, $\mathfrak{g} \oplus \mathbb{C}\hat{d} \subset \widehat{\mathfrak{g}}$ is a \hat{d} -embedding of augmented symmetrizable Kac-Moody algebras. Also, it is easy to check that $\hat{V} \in \mathcal{I}_R$ and is an integrable $\widehat{\mathfrak{g}}$ -representation.

For the representations $\hat{V} \in \mathcal{I}_A$, we refine Proposition 2 below to obtain the character of $\text{Spin}_0(\hat{V})$ in terms of the character of V .

Proposition 5. *Let V be a finite dimensional orthogonal \mathfrak{g} -representation. Let T be the set of all weights of V and m_β the multiplicity of a weight $\beta \in T$ so that the character of V can be written as:*

$$\text{Char } V = \sum_{\beta(d_1) > 0} m_\beta (e^\beta + e^{-\beta}) + m_0.$$

Then Spin_0 of the affinized $\widehat{\mathfrak{g}}$ -representation \hat{V} has the character:

$$\text{Char } \text{Spin}_0(\hat{V}) = e^{\nu + c\Lambda_0} \prod_{\beta(d_1) > 0} (1 + e^{-\beta})^{m_\beta} \prod_{\substack{k > 0 \\ \beta \in T}} (1 + e^{-\beta - k\delta})^{m_\beta},$$

where $\nu = \frac{1}{2} \sum_{\beta(d_1) > 0} m_\beta \beta$ and $c = \frac{1}{2} \sum_{\beta(d_1) > 0} m_\beta \beta(\theta^\vee)^2$, called the level of $\text{Spin}_0(\hat{V})$. Here θ is the highest root of \mathfrak{g} .

1.5.2. Factorization Theorems for affine Lie algebras. If we restrict an affinized $\widehat{\mathfrak{g}}$ -representation $\text{Spin}_0(\hat{V})$ to $\mathfrak{g} \oplus \mathbb{C}\hat{d}$ and apply Proposition 5, we obtain:

Proposition 6. $\text{Spin}_0(\hat{V})$, when restricted from $\widehat{\mathfrak{g}}$ to $\mathfrak{g} \oplus \mathbb{C}\hat{d}$, factors into an infinite tensor product:

$$\text{Spin}_0(\hat{V}) \downarrow_{\mathfrak{g} \oplus \mathbb{C}\hat{d}}^{\widehat{\mathfrak{g}}} \cong \text{Spin}_0(V) \otimes \wedge^\bullet(tV) \otimes \wedge^\bullet(t^2V) \otimes \cdots,$$

Remark. The $\mathfrak{g} \oplus \mathbb{C}\hat{d}$ -representation $U_k := \wedge^\bullet(t^k V)$ contains a canonical one-dimensional representation $\mathbb{C}1 = \wedge^0 t^k V$. The infinite tensor product above is the direct limit of the maps:

$$\begin{aligned} U_0 \otimes U_1 \otimes \cdots \otimes U_k &\rightarrow U_0 \otimes U_1 \otimes \cdots \otimes U_k \otimes U_{(k+1)} \\ u_0 \otimes u_1 \otimes \cdots \otimes u_k &\mapsto u_0 \otimes u_1 \otimes \cdots \otimes u_k \otimes 1 \end{aligned}$$

where $U_0 := \text{Spin}_0(V)$.

We now work out Theorems 1 and 2 for Affine Lie algebras.

Proposition 7. *The $\widehat{\mathfrak{g}}$ -representation $V(\hat{\rho})$, when restricted to $\mathfrak{g} \oplus \mathbb{C}\hat{d}$, factors into an infinite tensor product:*

$$V(\hat{\rho}) \downarrow_{\mathfrak{g} \oplus \mathbb{C}\hat{d}}^{\widehat{\mathfrak{g}}} \cong V(\rho) \otimes \wedge^\bullet(t\mathfrak{g}) \otimes \wedge^\bullet(t^2\mathfrak{g}) \otimes \cdots.$$

Proposition 8. *If we let:*

$$V(\hat{\mu}) \downarrow_{\mathfrak{g} \oplus \mathbb{C}\hat{d}}^{\widehat{\mathfrak{g}}} \cong \bigoplus_i V(\mu_i),$$

then:

$$V(2\hat{\mu} + \hat{\rho}) \downarrow_{\mathfrak{g} \oplus \mathbb{C}\hat{d}}^{\widehat{\mathfrak{g}}} \cong \left(\bigoplus_i V(2\mu_i + \rho) \right) \otimes \wedge^\bullet t\mathfrak{g} \otimes \wedge^\bullet t^2\mathfrak{g} \otimes \cdots.$$

1.6. Classification of coprimary representations. Motivated by Proposition 6, we ask: For which representations V is $\text{Spin}_0(\hat{V})$ irreducible?

Definition. A \mathfrak{g} -representation V is *coprimary* if $\text{Spin}_0(V)$ is irreducible.

Panyushev [P] gives a complete list of coprimary representations V of a simple Lie algebra \mathfrak{g} and deduces the classification for a semi-simple Lie algebra.

Proposition 9. *Let V be an orthogonal representation of a finite dimensional simple Lie algebra \mathfrak{g} . Then V is coprimary i.e. $\text{Spin}_0(V)$ is irreducible if and only if V is itself irreducible and is one of the following :*

- (1) $V(\theta)$, for all \mathfrak{g} where $\text{Spin}_0(V) = V(\rho)$;
- (2) $V(\theta_s)$, for $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{f}_4\}$ where $\text{Spin}_0(V) = V(\rho_s)$;
- (3) $V(2\theta_s)$, for $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$ ($n \geq 1$) where $\text{Spin}_0(V) = V(2\rho_s + \rho)$;

where θ_s = highest short root of \mathfrak{g} .

Remark. For $n = 1$, we have $\mathfrak{so}_{2n+1}\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$, and we take $\theta_s := \theta$.

The classification of coprimary affinized representations is as follows:

Proposition 10. *For a representation $\hat{V} \in \mathcal{I}_A$ of $\widehat{\mathfrak{g}}$ obtained from a representation V of a simple Lie algebra \mathfrak{g} ,*

$$\hat{V} \text{ is coprimary} \iff \left(\begin{array}{l} V \text{ is coprimary and belongs to} \\ \text{cases 1 or 2 of Proposition 9} \end{array} \right)$$

Panyushev proves the irreducibility of $\text{Spin}_0(V)$ using the Weyl denominator identity for the Langlands dual of \mathfrak{g} , and analogously we can prove the irreducibility of $\text{Spin}_0(\hat{V})$ in Proposition 10 using the Weyl denominator identity for the Langlands dual of $\widehat{\mathfrak{g}}$, a (possibly twisted) affine Lie algebra.

2. GENERAL SPIN CONSTRUCTION FOR AUGMENTED SYMMETRIZABLE KAC-MOODY ALGEBRAS

Next we give the construction of Spin_0 for representations of augmented symmetrizable Kac-Moody algebras. This surprisingly delicate matter has been briefly studied by Kac and Peterson [KP] and Pressley and Segal [PS, Chapter 12]. We provide here a different and a more detailed presentation. Also, we will do this in a more general setting which is compatible with restriction of representations.

Let V be a vector space with basis $\{e_i : i \in I\}$ where the index set can be finite, $I = \{m, \dots, 1, 0, -1, \dots, -m\}$ or $\{m, \dots, 1, -1, \dots, -m\}$; or infinite, $I = \mathbb{Z}$ or $\mathbb{Z} \setminus \{0\}$. Define a symmetric bilinear form on V by $Q(e_i, e_j) := \delta_{i, -j}$.

2.1. Finite dimensional case. First, let V be finite dimensional (I is finite). The *orthogonal Lie algebra* $\mathfrak{so}(V)$ is defined to consist of matrices which are skew-symmetric with respect to the anti-diagonal i.e.

$$\mathfrak{so}(V) := \{A = (a_{i,j})_{i,j \in I} : a_{i,j} = -a_{-j,-i}\}.$$

Thus, $\mathfrak{so}(V)$ has a basis $\{Z_{i,j} := E_{i,j} - E_{-j,-i} : i, j \in I, i > -j\}$ where $E_{i,j}$ are the coordinate matrices. We define the *Clifford algebra* $C(V, Q)$ as the associative algebra with 1 generated by all $v \in V$ with defining relations $e_{-i}e_j = -e_j e_{-i} + 2\delta_{ij} \forall i, j \in I$. There is an embedding of Lie algebras defined by :

$$\begin{aligned} \phi_F : \mathfrak{so}(V) &\longrightarrow C(V, Q) \\ Z_{i,j} &\longmapsto \frac{1}{4}(e_{-i}e_j - e_j e_{-i}). \end{aligned}$$

Now, the Clifford algebra has an action on a *wedge space* $\text{Spin}(V) := \wedge^\bullet V^+$, which on generators $\{e_i : i \in I\}$ of $C(V, Q)$ is as follows : Define $I^+ = \{i \in I : i > 0\}$. Then $\wedge^\bullet V^+$ has a basis $\{e_J := e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_k}\}$ for $0 \leq k \leq |I^+|$ and $J := \{j_1 > j_2 > \dots > j_k\} \subset I^+$. Here $|A| := \#(A)$. For $i \in I^+$ define:

$$e_i(e_J) := e_i \wedge e_J, \quad J \neq \emptyset; \quad e_{-i}(e_J) := \begin{cases} \epsilon(i, J) e_{(J \setminus \{i\})} & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases},$$

where $\epsilon(i, J) := 2(-1)^{|\{j \in J : j > i\}|}$. Also, $1(e_J) := e_J, e_i(e_\emptyset) = 1 = e_i$ and $e_0(e_J) := (-1)^{|J|} e_J$ if $0 \in I$. Finally, due to the embedding ϕ_F defined earlier, this action of $C(V, Q)$ induces an action of $\mathfrak{so}(V)$ on $\wedge^\bullet V^+$ which is called *Spin representation* of orthogonal Lie algebra $\mathfrak{so}(V)$.

As described in Section 1.3, for an orthogonal \mathfrak{g} -representation $V, \mathfrak{g} \subset \mathfrak{so}(V)$ and \mathfrak{g} acts on $\text{Spin}(V)$ by restriction. It is easy to find the character of $\text{Spin}(V)$ (see [P]) as a \mathfrak{g} representation. It turns out that if m_0 is the dimension of the zero weight space of V then $\text{Spin}(V)$ is isomorphic to the direct sum of $2^{\lfloor m_0/2 \rfloor}$ copies of another \mathfrak{g} -representation which is defined as $\text{Spin}_0(V)$. The character of $\text{Spin}_0(V)$ is given in Proposition 2.

2.2. General case. Now, let V be infinite-dimensional (I is \mathbb{Z} or $\mathbb{Z} - \{0\}$) with only finite linear combinations of $\{e_i : i \in I\}$ allowed. First, we naively extend the above definitions with the following modifications: The orthogonal Lie algebra, now denoted by $\mathfrak{so}_\infty(V)$, consist of skew-symmetric matrices with respect to anti-diagonal (as before) which have only finite number of non-zero entries in each column (skew-symmetry implies the same on the rows too), so that $\mathfrak{so}_\infty(V)$ is closed under commutator. Clifford algebra is allowed to have *infinite* sums of finite products of $\{e_i : i \in I\}$. The map ϕ_F defined in the finite dimensional case is still an embedding of Lie algebras. The infinite wedge $\text{Spin}(V) := \wedge^\bullet V^+$ is now an infinite dimensional vector space consisting of *finite* linear combinations of finite wedges of $\{e_i : i \in I^+\}$.

The action of $\{e_i : i \in I\}$ on $\wedge^\bullet V^+$, as defined in the finite case, does not extend to Clifford algebra nor induce an action of $\mathfrak{so}_\infty(V)$. For example, for $Y := \sum_{i \in \mathbb{Z}_{>0}} Z_{-i, i+1} \in \mathfrak{so}_\infty(V)$, $\phi_F(Y) = \sum_{i \in \mathbb{Z}_{>0}} \frac{e_i e_{i+1}}{2}$ does not act on $1 \in \wedge^\bullet V^+$ as it leads to an infinite sum. Also, for $H := \sum_{i \in I^+} Z_{i, i}$, an infinite diagonal matrix in $\mathfrak{so}_\infty(V)$, $\phi_F(H) = \sum_{i \in \mathbb{Z}_{>0}} \frac{1 - e_i e_{-i}}{2} \in C(V, Q)$ does not act on $1 \in \wedge^\bullet V^+$ as $\phi_F(Z_{i, i})(1) = 1/2$.

In order to resolve these two issues, next we suitably modify $\mathfrak{so}_\infty(V)$ and ϕ_F and define a smaller Lie algebra $\mathfrak{so}(V)$ and a map ϕ so that the image of $\mathfrak{so}(V)$ under ϕ does act on $\wedge^\bullet V^+$.

The Lie algebra $\mathfrak{so}(V)$ consists of matrices $A = (a_{i,j})_{i,j \in I}$ such that:

- (1) A is skew-symmetric with respect to the anti-diagonal: $a_{i,j} = -a_{-j,-i}$.

(2) Each column $(a_{ij})_{i \in I}$ has finitely many non-zero entries.

(3) The blocks $(a_{i,-j})_{i,j>0}$ and $(a_{-i,j})_{i,j>0}$ have finitely many non-zero entries.

Define the map:

$$\begin{aligned} \phi : \mathfrak{so}(V) &\longrightarrow C(V, Q) \\ Z_{i,j} &\longmapsto -\frac{1}{2}e_j e_{-i}. \end{aligned}$$

Now referring to the matrices Y and H defined earlier, note that the matrix $Y \in \mathfrak{so}_\infty(V)$ does not belong to $\mathfrak{so}(V)$ and even though H belongs to $\mathfrak{so}(V)$, $\phi(H)$ does act on $1 \in \wedge^\bullet V^+$. Further, we can verify that $\text{image}(\phi) \subset C(V, Q)$ does act on $\wedge^\bullet V^+$. In exchange of getting the action it turns out that the map ϕ is *not* a Lie algebra map and $\text{image}(\phi)$ is not closed under brackets in $C(V, Q)$. But the central extension of the image(ϕ):

$$\tilde{\mathfrak{so}}(V) := \{\phi(A) : A \in \mathfrak{so}(V)\} \oplus \mathbb{C}1 \subset C(V, Q),$$

is a Lie algebra which also acts on $\wedge^\bullet V^+$ (as 1 acts by identity). The Lie algebra $\tilde{\mathfrak{so}}(V)$ is a central extension of $\mathfrak{so}(V)$ by one dimensional center $\mathbb{C}1$ due to the following exact sequence of Lie algebra maps :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \tilde{\mathfrak{so}}(V) & \xrightarrow{\pi} & \mathfrak{so}(V) & \longrightarrow & 0 \\ & & 1 & \longmapsto & 1 & \longmapsto & 0 & & \\ & & & & e_j e_{-i} & \longmapsto & -2Z_{i,j}. & & \end{array}$$

This can be verified using the following commutator relations in $\tilde{\mathfrak{so}}(V)$ and $\mathfrak{so}(V)$.

$$\begin{aligned} [e_j e_{-i}, e_s e_{-r}] &= 2\delta_{i,s} e_j e_{-r} - 2\delta_{i,-r} e_j e_s + 2\delta_{j,-s} e_{-r} e_{-i} - 2\delta_{j,r} e_s e_{-i} \\ [2Z_{i,j}, 2Z_{r,s}] &= -4\delta_{i,s} Z_{r,j} + 4\delta_{i,-r} Z_{-s,j} - 4\delta_{j,-s} Z_{i,-r} + 4\delta_{j,r} Z_{i,s}. \end{aligned}$$

We can prove that this extension does not split when V is infinite dimensional. Thus $\tilde{\mathfrak{so}}(V)$ -representation $\wedge^\bullet V^+$ which we call *Spin representation* does not induce an action of $\mathfrak{so}(V)$. Therefore, the orthogonal Lie algebra $\mathfrak{so}(V)$, when V is infinite dimensional, does *not* have a Spin representation, but its central extension $\tilde{\mathfrak{so}}(V)$ does.

The above construction can also be carried out when V is finite dimensional. There the extension splits as $(\text{image}(\phi_F) \oplus \mathbb{C}1)$ because ϕ_F is an embedding and $\pi \circ \phi_F = \text{I}_{\mathfrak{so}(V)}$. Thus, $\tilde{\mathfrak{so}}(V)$ -representation $\wedge^\bullet V^+$ does induce an action of $\mathfrak{so}(V)$ and the resulting representation coincides with the Spin representation of $\mathfrak{so}(V)$ defined earlier. Thus, the above construction is the general construction of $\text{Spin}(V)$ for a finite or infinite dimensional vector space V .

Now let \mathfrak{g} be an augmented symmetrizable Kac-Moody algebra with distinguished element d and V a d -finite orthogonal \mathfrak{g} -representation (as in §1.2). Orthogonality and d -finiteness of V lead to a map $\tilde{\mathfrak{g}} \rightarrow \mathfrak{so}(V)$. Once we have the map $\mathfrak{g} \rightarrow \mathfrak{so}(V)$, using $\pi : \tilde{\mathfrak{so}}(V) \rightarrow \mathfrak{so}(V)$ defined earlier, we get an induced map $\mathfrak{g} \rightarrow \tilde{\mathfrak{so}}(V)$ for any augmented symmetrizable Kac-Moody algebra due to the following lemma.

Lemma 1. *Let \mathfrak{g} be an augmented symmetrizable Kac-Moody algebra with Cartan subalgebra \mathfrak{h} . Fix a complementary subspace \mathfrak{h}'' to $\mathfrak{h}' := \bigoplus_{i=1}^n \mathbb{C}\alpha_i^\vee$ in \mathfrak{h} . Thus :*

$$\mathfrak{g} = \mathfrak{h}' \oplus \mathfrak{h}'' \oplus \mathfrak{g}_R,$$

where \mathfrak{g}_R is the space spanned by all roots. Fix $\psi \in (\mathfrak{h}'' \oplus \mathfrak{g}_R)^*$ such that $\psi(\mathfrak{g}_R) = 0$. Then for any Lie algebra map $\sigma : \mathfrak{g} \rightarrow \mathfrak{so}(V)$ there exists a unique lifting $\tilde{\sigma} : \mathfrak{g} \rightarrow \tilde{\mathfrak{so}}(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\sigma} & \mathfrak{so}(V) \\ \tilde{\sigma} \searrow & & \uparrow \pi \\ & & \tilde{\mathfrak{so}}(V) \end{array}$$

and $\tilde{\sigma} = \phi \circ \sigma + \psi$ on $\mathfrak{h}'' \oplus \mathfrak{g}_R$. Recall that $\tilde{\mathfrak{so}}(V)$ was defined as $\phi(\mathfrak{so}(V)) \oplus \mathbb{C}1$.

Now since $\tilde{\mathfrak{so}}(V)$ acts on $\text{Spin}(V)$, that is:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\sigma} & \mathfrak{so}(V) \\ \tilde{\sigma} \searrow & & \uparrow \pi \\ & & \tilde{\mathfrak{so}}(V) \longrightarrow \text{End}_{\mathbb{C}} \text{Spin}(V), \end{array}$$

for a given orthogonal d -finite \mathfrak{g} -representation V , we can define the \mathfrak{g} -representation $\text{Spin}(V)$. In §1.3, we defined $\text{Spin}_0(V)$ and some basic properties of $\text{Spin}(V)$ and $\text{Spin}_0(V)$ are listed.

3. PROOFS

3.1. Proof of Lemma 1. Let $X_{\pm i}, i = 1 \cdots n$ be the simple root vectors of \mathfrak{g} and $\{d_1, d_2, \dots, d_l\}$ be a basis of \mathfrak{h}'' . Then, $X_{\pm i}$'s and d_j 's generate \mathfrak{g} as a Lie algebra.

Note that by the commutativity of the diagram and due to the map ψ , the map $\tilde{\sigma}$ is uniquely defined on the generators of \mathfrak{g} . Now, to be able to extend this map $\tilde{\sigma}$ to whole of \mathfrak{g} , we need that $\tilde{\sigma}(X_{\pm i})$ and $\tilde{\sigma}(d_j)$ in $\tilde{\mathfrak{so}}(V)$ satisfy the defining bracket relations of \mathfrak{g} . But since $\sigma(X_{\pm i})$ and $\sigma(d_j)$ in $\mathfrak{so}(V)$ satisfy the defining bracket relations of \mathfrak{g} (as σ is a Lie algebra map), and $\pi : \tilde{\mathfrak{so}}(V) \longrightarrow \mathfrak{so}(V)$ is a Lie algebra map mapping $\tilde{\sigma}(X_{\pm i}), \tilde{\sigma}(d_j)$ to $\sigma(X_{\pm i}), \sigma(d_j)$ in $\mathfrak{so}(V)$, we can prove that $\tilde{\sigma}(X_{\pm i}), \tilde{\sigma}(d_j)$ also satisfy each defining bracket relation of \mathfrak{g} up to a constant because $\ker(\pi) = \mathbb{C}$. We show that this constant is zero for each relation.

Let $\hat{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}$ be the cartan subalgebras of $\mathfrak{so}(V)$ and $\tilde{\mathfrak{so}}(V)$ respectively so that $\tilde{\mathfrak{h}} = \pi^{-1}(\hat{\mathfrak{h}})$. Then the constant term in $\tilde{\sigma}(X_{\pm i})$, when expressed in standard basis of $\tilde{\mathfrak{so}}(V)$, is zero and $\tilde{\sigma}(d_j) \in \tilde{\mathfrak{h}}$ as $\sigma(d_j) \in \hat{\mathfrak{h}}$. Set $\tilde{\sigma}(\alpha_i^\vee) = [\tilde{\sigma}(X_i), \tilde{\sigma}(X_{-i})]$ for $i = 1, \dots, n$. Then clearly, $\tilde{\sigma}(\alpha_i^\vee) \in \tilde{\mathfrak{h}}$ for all i . This defines $\tilde{\sigma}$ from \mathfrak{h} into $\tilde{\mathfrak{h}}$ (may not be injective). For any $H \in \mathfrak{h}$ we can easily conclude that $[\tilde{\sigma}(H), \tilde{\sigma}(X_{\pm i}) - \alpha_i(H)\tilde{\sigma}(X_{\pm i})]$ is a constant. This constant must be zero because the constant term in $\tilde{\sigma}(X_{\pm i})$ is zero and $\tilde{\sigma}(H) \in \tilde{\mathfrak{h}}$. Now for $i \neq j$, $X := [\tilde{\sigma}(X_i), \tilde{\sigma}(X_{-j})]$ must be constant and for all $H \in \mathfrak{h}$, $\tilde{\sigma}(H)$ acts diagonally on X with eigenvalue $(\alpha_i - \alpha_j)(H)$. This implies that $X = 0$. Similarly, the generators satisfy the last bracket relation also (see §1.1).

3.2. Proof of Propositions 1-2. First, we compute the character of $\text{Spin}(V)$ as a $\tilde{\mathfrak{so}}(V)$ -representation. As in §2, let V be the vector space with basis $\{e_i : i \in I\}$. We can check that $\tilde{\mathfrak{h}} := \bigoplus_{i \in I^+} \mathbb{C}e_i e_{-i} \oplus \mathbb{C}1$ is a Cartan subalgebra of $\tilde{\mathfrak{so}}(V)$. Consider the dual basis of $\{1, \frac{e_j e_{-j}}{2} : j \in I^+\}$ i.e. $\tilde{L}_i \in \tilde{\mathfrak{h}}^*$ defined as :

$$\begin{aligned} \tilde{L}_0(1) &= 1 & \tilde{L}_0(\frac{e_j e_{-j}}{2}) &= 0 & j &\in I^+ \\ \tilde{L}_i(1) &= 0 & \tilde{L}_i(\frac{e_j e_{-j}}{2}) &= \delta_{i,j} & i, j &\in I^+. \end{aligned}$$

A basis of weight vectors of $\wedge^\bullet V^+$ is $\{e_J : J \subset I^+\}$, where $e_J := e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}$, for $J = \{j_1, j_2, \dots, j_k\}$. Observe that :

$$\begin{aligned} 1(e_J) &= e_J \\ \frac{e_j e_{-j}}{2}(e_J) &= \begin{cases} e_J & \text{if } j \in J \\ 0 & \text{if } j \notin J. \end{cases} \end{aligned}$$

So, e_J has the weight $\tilde{L}_0 + \sum_{j \in J} \tilde{L}_j$. Therefore,

$$\begin{aligned} \text{Char Spin}(V) &= \text{Char}(\wedge^\bullet V^+) = \sum_{J \subset I^+} e^{\tilde{L}_0 + \sum_{j \in J} \tilde{L}_j}, \\ &= e^{\tilde{L}_0} \prod_{i \in I^+} (1 + e^{\tilde{L}_i}). \end{aligned}$$

as $\tilde{\mathfrak{so}}(V)$ -representation. Due to the map, $\mathfrak{g} \xrightarrow{\tilde{\sigma}} \tilde{\mathfrak{so}}(V)$, (described in §2 Lemma 1), $\text{Spin}(V)$ becomes a \mathfrak{g} -representation. Then, $\tilde{L}_i \circ \tilde{\sigma}$, $i \in (I^+ \cup \{0\})$, are the weights of $\text{Spin}(V)$ as a \mathfrak{g} -representation. Due to the commutativity of the diagram in lemma 1 we can show that $\tilde{L}_i \circ \tilde{\sigma} = -L_i \circ \sigma$, $i \in I^+$, where L_i 's are the weights of the defining representation V of $\mathfrak{so}(V)$. Let $\beta_i := L_i \circ \sigma$, $i \in I^+$ and $\Lambda := \tilde{L}_0 \circ \tilde{\sigma}$. Thus, as a \mathfrak{g} -representation,

$$\text{Char Spin}(V) = e^\Lambda \prod_{i \in I^+} (1 + e^{-\beta_i}).$$

Without loss of generality, we may assume $\beta_i(d) \geq 0$ for all $i \in I^+$. When V is finite dimensional (see [P]), $\Lambda = \frac{1}{2} \sum_{i \in I^+} \beta_i$.

3.2.1. *Proof of Proposition 1.* Now we prove parts (1) - (7) of Proposition 1.

Proof of Proposition 1(1). We show the following:

V is d -finite

$\Rightarrow \text{Spin}(V)$ is d -finite and the set $\{\gamma(d) : \gamma \text{ is a weight of } \text{Spin}(V)\}$ is bounded above.

$\Rightarrow \text{Spin}(V) \in \mathcal{O}_{\text{weak}}$.

Let V be d -finite. Result is obvious if V is finite dimensional. So, let V be a infinite dimensional so that $I = \mathbb{Z}$, $I^+ = \mathbb{Z}_{>0}$ and the set of positive weights are $\{\beta_i : i \in \mathbb{Z}_{>0}\}$. Any weight of $\text{Spin}(V)$ is of the form $\Lambda - a$ where $a = \sum_{i \in \mathbb{Z}_{>0}} a_i \beta_i$ for $a_i = 0$ or 1 for $i \in \mathbb{Z}_{>0}$ where $a_i = 0$ for all but finitely many i 's. Let's call such a sequence, $(a_i)_{i \in \mathbb{Z}_{>0}}$, an (a) -sequence.

For d -finiteness of $\text{Spin}(V)$, it's enough to show that the above character when restricted to $\mathbb{C}d$ has finite coefficients. This is equivalent to: For each N , there are finitely many (a) -sequences such that $(\Lambda - a)(d) = N$. Define

$$L_k := \{i \in \mathbb{Z}_{>0} : \beta_i(d) = k\},$$

L_k is a finite set due to d -finiteness of V .

$$\begin{aligned} (\Lambda - a)(d) &= N \\ \Rightarrow \Lambda(d) - N &= \sum_{i \in \mathbb{Z}_{>0}} a_i \beta_i(d) \\ \Rightarrow \Lambda(d) - N &= \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\sum_{\beta_i(d)=k} a_i \right) k. \\ \Rightarrow \Lambda(d) - N &= \sum_{k \in \mathbb{Z}_{\geq 0}} \left(\sum_{i \in L_k} a_i \right) k. \\ (1) \quad \Rightarrow \Lambda(d) - N &= \sum_{k \in \mathbb{Z}_{\geq 0}} b_k k, \end{aligned}$$

where $b_k := \sum_{i \in L_k} a_i$. In the above equation $k \in \mathbb{Z}_{\geq 0}$ because $\beta_i(d) \geq 0$ for all i and the sum is a finite sum as $a_i \neq 0$ only for finitely many i 's. Clearly, b_k is finite. Thus each (a) -sequence $(a_i)_{i \in \mathbb{Z}_{>0}}$ satisfying equation (1) gives a non-negative integral partition of $\Lambda(d) - N$ where each non-negative integer is repeated b_k times. Conversely, for every such partition given by $(b_k)_{k \in \mathbb{Z}_{\geq 0}}$ with all but finite number of b_k to be zero, there exists only finitely many (a) -sequences, $(a_i)_{i \in \mathbb{Z}_{>0}}$ such that $b_k = \sum_{i \in L_k} a_i$. Since there are finitely many such partitions of $\Lambda(d) - N$, there are finitely many (a) -sequences satisfying equation (1). Thus $\text{Spin}(V)$ is d -finite.

Finally, $\{\gamma(d) : \gamma \text{ is a weight of } \text{Spin}(V)\}$ is bounded above by $\Lambda(d)$ as $\gamma = \Lambda - a$ for $a = \sum_{i \in \mathbb{Z}_{>0}} a_i \beta_i$ and $\beta_i(d) \geq 0$. That proves the first implication.

To prove $\text{Spin}(V) \in \mathcal{O}_{\text{weak}}$, we show that if W is a d -finite representation, such that $\{\gamma(d) : \gamma \text{ is a weight of } W\}$ is bounded above then $W \in \mathcal{O}_{\text{weak}}$. First of all weight spaces of W are finite dimensional by d -finiteness of W . So, let $\mathcal{P}(W)$ denote the set of all weights of W . For $\beta \in \mathcal{P}(W)$, let $\gamma \in S_\beta := \{\gamma \in \mathcal{P}(W) : \beta \leq \gamma\}$. That is, $\gamma - \beta = \sum c_\alpha \alpha$, for some $c_\alpha \in \mathbb{Z}_{\geq 0}$ where the sum is over simple positive roots of \mathfrak{g} . This implies $\gamma(d) - \beta(d) \geq 0$ as $\alpha(d) > 0$ by definition of d . As $\gamma(d)$ is bounded above, $K \leq \gamma(d) \leq N$ where $K = \beta(d)$. So S_β is finite, otherwise $\bigoplus_{K \leq \gamma(d) \leq N} W^{(\gamma)}$ will be infinite dimensional contradicting the d -finiteness of W (see §1.2). Now the set of all maximal weights of the non-empty finite set S_β is non-empty and intersects M_W non-trivially. Thus $W \in \mathcal{O}_{\text{weak}}$. \square

Proof of Proposition 1(2). The result is obvious if V is finite dimensional. Assume that V is infinite dimensional. Let X be a positive or negative root vector of \mathfrak{g} . We will show that X acts locally nilpotently on $\text{Spin}(V)$ if it acts locally nilpotently on $\mathfrak{so}(V)$ (see §2).

First assume X to be a positive root vector. We denote the matrix of the action of X on the representation V given by the map $\mathfrak{g} \rightarrow \mathfrak{so}(V)$ by X only. Recall, that $\mathfrak{so}(V)$ is defined with respect to a polarization of $V = V^+ \oplus V^-$. Fix an ordered basis $\{\dots > e_2 > e_1 > e_{-1} > e_{-2} > \dots\}$ of V , where $V^\pm = \bigoplus_{i \in \mathbb{Z}^\pm} \mathbb{C}e_i$. We may assume that X is a strictly upper triangular matrix in $\mathfrak{so}(V)$ with respect to above basis. We can write $X = Z + F$ such that $Z(V^\pm) \subset V^\pm$, $F(V^\pm) \subset V^\mp$. Then by definition of $\mathfrak{so}(V)$, $F(V)$ is finite dimensional. Since X is a upper triangular, $F(V^+) = \{0\}$. Thus, let $F := \sum_{r,t} b_{r,t} Z_{r,-t}$, a finite sum, where $\{Z_{i,j}\}$ forms a basis of $\mathfrak{so}(V)$ defined in §2. Referring to Lemma 1 in §2.2, the image of X , say \tilde{X} , in $\tilde{\mathfrak{so}}(V)$ can be written as: $\tilde{X} = \tilde{Z} + \tilde{F}$, where $\tilde{Z} = \phi(Z)$ and $\tilde{F} = \phi(F)$ in $\tilde{\mathfrak{so}}(V)$ (see §2.2). Here $\tilde{F} = -1/2 \sum_{r,t} b_{r,t} e_{-t} e_{-r}$. Now action of \tilde{Z} and \tilde{F} on $\text{Spin}(V) := \wedge^\bullet V^+$ is defined as:

$$\tilde{Z}(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}) := \hat{Z}e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} + e_{i_1} \wedge \hat{Z}e_{i_2} \wedge \dots \wedge e_{i_k} + \dots$$

where \hat{Z} denotes the transpose of Z . Thus,

$$\tilde{Z}^p(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}) = \sum_{p_1 + \dots + p_k = p} \frac{p!}{p_1! \dots p_k!} \hat{Z}^{p_1} e_{i_1} \wedge \hat{Z}^{p_2} e_{i_2} \wedge \dots \wedge \hat{Z}^{p_k} e_{i_k}.$$

and

$$\tilde{F}(e_{i_1} \wedge \dots \wedge e_r \wedge \dots \wedge e_t \wedge \dots \wedge e_{i_k}) := \sum_{r,t} c_{r,t} e_{i_1} \wedge \dots \wedge \hat{e}_r \wedge \dots \wedge \hat{e}_t \wedge \dots \wedge e_{i_k}.$$

where \hat{e}_i denotes the absence of e_i and $c_{r,t} = \pm 2b_{r,t}$ depending on positions of e_r and e_t . Also, $c_{r,t} = 0$ if e_r and/or e_t do not occur in the wedge. Let

$$V_m := \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \dots \oplus \mathbb{C}e_m.$$

for $m \in \mathbb{Z}_{>0}$ and let

$$\hat{Z}^\bullet V_m := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \hat{Z}^k(V_m).$$

By definition of Z , above is finite direct sum as $\hat{Z}^k(V_m) = 0$ for large k , say q .

Lemma 2. *For any given $K \in \mathbb{Z}_{>0}$ and a matrix Z in $\mathfrak{so}(V)$ satisfying $Z(V^\pm) \subset V^\pm$ and $Z^q(V_m) = 0$ for some q , $\tilde{Z}^P[\wedge^k(\hat{Z}^\bullet V_m)] = 0$ for all $k \leq K$ and $P = K(q-1) + 1$.*

Proof. Let $a := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \in \wedge^k(\hat{Z}^\bullet V_m)$.

$$\tilde{Z}^P(a) = \sum_{q_1 + \cdots + q_k = P} \frac{P!}{q_1! \cdots q_k!} \hat{Z}^{q_1} e_{i_1} \wedge \hat{Z}^{q_2} e_{i_2} \wedge \cdots \wedge \hat{Z}^{q_k} e_{i_k}.$$

Since, for $1 \leq s \leq k$, $e_{i_s} \in \hat{Z}^\bullet V_m \Rightarrow \hat{Z}^{q_s}(e_{i_s}) \in \hat{Z}^\bullet(\hat{Z}^{q_s} V_m)$. Now, if $q_s \leq q-1$ for all s then $P = \sum_{s=1}^k q_s \leq k(q-1) \leq K(q-1) = P-1$. So, $P \leq P-1 \Rightarrow \Leftarrow$. Thus, $q_s \geq q$ for some s . And, $\hat{Z}^{q_s}(e_{i_s}) \in \hat{Z}^\bullet(\hat{Z}^{q_s} V_m) = 0$ for some s meaning $\tilde{Z}^P(a) = 0$ for all $a \in \wedge^k(\hat{Z}^\bullet V_m)$ since a is arbitrary. Hence, $\tilde{Z}^P[\wedge^k(\hat{Z}^\bullet V_m)] = 0$ which is Lemma 2. \square

We will prove that for a given $s \in \mathbb{Z}_{>0}$, we can find N , depending on s such that:

$$(2) \quad (\tilde{Z} + \tilde{F})^N(\wedge^s V_m) = 0,$$

which will prove that $\tilde{X} = \tilde{Z} + \tilde{F}$ acts nilpotently on $\text{Spin}(V) = \wedge^\bullet V^+$.

Set $l := \lfloor \frac{s}{2} \rfloor$, $P := s(q-1) + 1$ and $N := (l+1)(P-1) + l + 1$. Consider:

$$(\tilde{Z} + \tilde{F})^N(\wedge^s V_m) = \bigoplus_{k=0, N} \left(\bigoplus_{\sum P_i = N-k} \tilde{Z}^{P_1} \tilde{F} \tilde{Z}^{P_2} \cdots \tilde{F} \tilde{Z}^{P_{k+1}} \right) (\wedge^s V_m).$$

Verify that for any $p, k \in \mathbb{Z}_{>0}$, $\tilde{Z}^p(\wedge^k(\hat{Z}^\bullet V_m)) \subset \wedge^k(\hat{Z}^\bullet V_m)$ and $\tilde{F}(\wedge^k(\hat{Z}^\bullet V_m)) \subset \wedge^{k-2}(\hat{Z}^\bullet V_m)$. Thus,

$$U := \tilde{Z}^{P_1} \tilde{F} \tilde{Z}^{P_2} \cdots \tilde{F} \tilde{Z}^{P_{k+1}}(\wedge^s V_m) \subset \wedge^{s-2k}(\hat{Z}^\bullet V_m),$$

where right hand is defined as zero for $s < 2k$. Let $k \leq \lfloor \frac{s}{2} \rfloor = l$. Since $\sum_i P_i = N-k$, $P_i \geq P$ for some i for a similar reason as in Lemma 2. Let $P_j \geq P$. Now,

$$\tilde{F} \tilde{Z}^{P_{j+1}} \cdots \tilde{F} \tilde{Z}^{P_{k+1}}(\wedge^s V_m) \subset \wedge^{s-2(k-j+1)}(\hat{Z}^\bullet V_m).$$

By Lemma 2,

$$\begin{aligned} & \hat{Z}^{P_j}(\tilde{F} \tilde{Z}^{P_{j+1}} \cdots \tilde{F} \tilde{Z}^{P_{k+1}})(\wedge^s V_m) = 0 \\ \Rightarrow U &:= \tilde{Z}^{P_1} \tilde{F} \cdots \tilde{F} \hat{Z}^{P_j}(\tilde{F} \tilde{Z}^{P_{j+1}} \cdots \tilde{F} \tilde{Z}^{P_{k+1}})(\wedge^s V_m) = 0 \\ & \Rightarrow (\tilde{Z} + \tilde{F})^N(\wedge^s V_m) = 0 \end{aligned}$$

which proves $\tilde{X} = \tilde{Z} + \tilde{F}$ is locally nilpotent on $\text{Spin}(V) = \wedge^\bullet V^+$.

Note that we have not used the integrability of V yet. Now, for X a negative root vector of \mathfrak{g} , whose matrix corresponds to a strictly lower triangular matrix, the proof will require the integrability of V . Most of the analysis is same but is significantly different at few places.

As before $X = Z + F$. This time $F(V^-) = \{0\}$, $F := \sum_{r,t} b_{r,t} Z_{-r,t}$ and $\tilde{F} = -1/2 \sum_{r,t} b_{r,t} e_t e_r$.

$$\tilde{F}(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) := -\frac{1}{2} \sum_{r,t} b_{r,t} e_t \wedge e_r \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

It is enough to prove equation (2) for all $m \geq \text{Max}\{r, t : b_{r,t} \neq 0\}$. In the previous case, we got for free the condition that $\hat{Z}^q(V_m) = 0$ for some q , because \hat{Z} was a strictly lower triangular matrix with $\hat{Z}(V^\pm) \subset V^\pm$. In this case, we use the fact that X is a locally nilpotent matrix. So, there exists a q such that $(Z + F)^q(V_m^-) = 0$ where $V_m^- := \mathbb{C}e_{-1} \oplus \mathbb{C}e_{-2} \oplus \cdots \oplus \mathbb{C}e_{-m}$. Since X is skew-symmetric, this is equivalent to $(\hat{Z} + \hat{F})^q(V_m) = 0$ where \hat{F} is the transpose of F . Since, $F(V^-) = \{0\}$, $\hat{F}(V^+) = \{0\}$. In particular, $\hat{F}(V_m) = \{0\}$ which leads to $\hat{Z}^q(V_m) = \{0\}$ as required.

Let $\hat{Z}^\bullet V_m \subset V_R$, for some R depending on m . We modify $l := \lfloor \frac{R-s}{2} \rfloor$ $P := (s+2l)(q-1)+1$ to define $N := (l+1)(P-1)+l+1$ as before. Also,

$$\tilde{F}(\wedge^k(\hat{Z}^\bullet V_m)) \subset \wedge^{k+2}(\hat{Z}^\bullet V_m)$$

for all $k \in \mathbb{Z}_{>0}$ so that

$$U := \tilde{Z}^{P_1} \tilde{F} \tilde{Z}^{P_2} \dots \tilde{F} \tilde{Z}^{P_{k+1}} (\wedge^s V_m) \subset \wedge^{s+2k}(\hat{Z}^\bullet V_m),$$

and is zero if $s+2k > R$ and the proof goes through.

The fact about $\text{Spin}_0(V)$ follows from the character formula given before the proof of Proposition 1. \square

Proof of Proposition 1(3). Let $\{e_i : i \in I_1\}$ be the chosen basis of weight vectors of V_1 , and $\{e'_i : i \in I_2\}$ of V_2 . If at least one of m_1 or m_2 is even, $(V_1 \oplus V_2)^+ = V_1^+ \oplus V_2^+$. Define the map:

$$\begin{array}{ccc} \text{Spin}(V_1 \oplus V_2) & \xrightarrow{p} & \text{Spin}(V_1) \otimes \text{Spin}(V_2) \\ e_J \wedge e'_K & \mapsto & e_J \otimes e'_K \\ 1 & \mapsto & 1 \otimes 1 \end{array}$$

for $J \subset I_1^+$ and $K \subset I_2^+$, where at least one of them is not empty and $e_\emptyset := 1$ and $e'_\emptyset := 1$. Here, p is an isomorphism of $\tilde{\mathfrak{so}}(V_1) \oplus \tilde{\mathfrak{so}}(V_2)$ -modules as it can be easily verified that $X \circ p = p \circ X$ for $X \in \tilde{\mathfrak{so}}(V_1)$ and $X \in \tilde{\mathfrak{so}}(V_2)$.

Now let both m_1 and m_2 be odd, so that the chosen bases of V_1 and V_2 are $\{\dots, e_2, e_1, e_0, e_{-1}, e_{-2}, \dots\}$ and $\{\dots, e'_2, e'_1, e'_0, e'_{-1}, e'_{-2}, \dots\}$ respectively. Define $u := (e_0 + ie'_0)/\sqrt{2}$, $v := (e_0 - ie'_0)/\sqrt{2}$ so that u, v are paired non-degenerately with respect to the bilinear form. Now, $(V_1 \oplus V_2)^+ = V_1^+ \oplus V_2^+ \oplus \mathbb{C}u$. Define the map,

$$\begin{array}{ccc} \text{Spin}(V_1) \otimes \text{Spin}(V_2) & \xrightarrow{p} & \text{Spin}^{\text{even}}(V_1 \oplus V_2) \\ e_{J_1} \otimes e'_{J_2} & \mapsto & \frac{(-1)^{t_1 t_2}}{i^{t_2}} e_{J_1} \wedge e'_{J_2} \wedge (1 - t + t \frac{u}{\sqrt{2}}) \\ 1 \otimes 1 & \mapsto & 1 \end{array}$$

where, $t_k := |J_k| \bmod 2$, $k = 1, 2$ and $t := (|J_1| + |J_2|) \bmod 2$ and $i := \sqrt{-1}$. Again, p is an isomorphism of $\tilde{\mathfrak{so}}(V_1) \oplus \tilde{\mathfrak{so}}(V_2)$ -modules as it can be easily verified that $X \circ p = p \circ X$ for $X \in \tilde{\mathfrak{so}}(V_1)$ and $X \in \tilde{\mathfrak{so}}(V_2)$. A similar isomorphism can be given for $\text{Spin}^{\text{odd}}(V_1 \oplus V_2)$. \square

Proof of Proposition 1(4). $\text{Spin}_0(V_1 \oplus V_2) \cong \text{Spin}_0(V_1) \otimes \text{Spin}_0(V_2)$.

Using Proposition 1(3), let at least one of m_1 or m_2 is even say, m_1 . So, let $m_1 = 2k_1$ and $m_2 = 2k_2 + \epsilon$, where $\epsilon = 0$ or 1. Now using Proposition 1(2) we get:

$$2^{k_1+k_2} \text{Spin}_0(V_1 \oplus V_2) \cong 2^{k_1} \text{Spin}_0(V_1) \otimes 2^{k_2} \text{Spin}_0(V_2),$$

which gives the desired result.

Now let $m_1 = 2k_1 + 1$ and $m_2 = 2k_2 + 1$ then as before we get:

$$2^{k_1+k_2+1-1} \text{Spin}_0(V_1 \oplus V_2) \cong 2^{k_1} \text{Spin}_0(V_1) \otimes 2^{k_2} \text{Spin}_0(V_2),$$

which again gives the desired result. \square

Remark. Proposition 1(4) is also true for an *infinite* direct sum $V := V_1 \oplus V_2 \oplus \dots$ if V is d -finite. Also, we define the infinite tensor product on the right hand side as: Let

$$U_k := \text{Spin}_0(V_1) \otimes \dots \otimes \text{Spin}_0(V_k) \otimes \text{Spin}_0(V_{k+1} \oplus V_{k+2} \oplus \dots).$$

There is an isomorphism $U_k \rightarrow U_{k+1}$ by Proposition 1(4). The infinite tensor product, $\text{Spin}_0(V_1) \otimes \text{Spin}_0(V_2) \otimes \dots$, is defined as direct limit of maps $U_k \rightarrow U_{k+1}$.

Proof of Proposition 1(5). Let W be root finite. So, there exists only finitely many weights of W outside the root cone C defined in §1.3 and W has finite dimensional weight spaces. Hence, for d -finiteness of W , it is enough to prove that the sets $S_k^\pm := \{\beta \in \mathcal{P} \cap C^\pm : \beta(d) = k\} \supset \{\beta \in \mathcal{P}(W) \cap C^\pm : \beta(d) = k\}$, is finite for each $k \in \mathbb{Z}^\pm$, where \mathcal{P} denotes the weight lattice of \mathfrak{g} , $\mathcal{P}(W)$ are the weights of W and $C^\pm := \left\{ \sum_{j=1}^n c_j \alpha_j : c_j \in \mathbb{R}^\pm \cup \{0\} \right\}$.

Let \mathcal{P}_R be the root lattice, $\mathcal{P}/\mathcal{P}_R = \{\mathcal{P}_1, \mathcal{P}_2, \dots\}$. There exists finitely many i 's for which $\mathcal{P}_i \cap C^+ \neq \{\}$, say $i = 1, \dots, m$. For $1 \leq i \leq m$, let $p_i \in \mathcal{P}_i \cap C^+$ be a coset representative of \mathcal{P}_i such that for each $\beta \in \mathcal{P}_i \cap C^+$ can be written as $\beta = p_i + \sum_{j=1}^n c_j \alpha_j$ with $c_j \in \mathbb{Z}_{\geq 0}$. Then

$$\{\beta \in \mathcal{P}_i \cap C^+ : \beta(d) = k\} = \left\{ p_i + \sum_{j=1}^n c_j \alpha_j : p_i(d) + \sum_{j=1}^n c_j \alpha_j(d) = k, c_j \in \mathbb{Z}_{\geq 0} \right\},$$

which is clearly a finite set using the definition of d . This leads to finiteness of S_k^+ . Similarly we prove that S_k^- is finite. \square

Proof of Proposition 1(6). Let V be d -finite and orthogonal. To prove:

$$V \text{ is root-finite} \Leftrightarrow \text{Spin}(V) \in \mathcal{O}.$$

(\Rightarrow)

Use the character formula given before the proof of Proposition 1,

$$\text{Char Spin}(V) = e^\Lambda \prod_{i \in I^+} (1 + e^{-\beta_i}).$$

Refer to Proof of Proposition 1(5) for the definitions of $C, C^+, C^-, \mathcal{P}, \mathcal{P}_R, p_j, \mathcal{P}_j, j = 1 \dots m$ and define the sets:

$$\begin{aligned} I_1 &:= \{i \in I^+ : \beta_i \in \mathcal{P} \cap C^+\}, \\ I_2 &:= \{i \in I^+ : \beta_i \in \mathcal{P} - C\}. \end{aligned}$$

Note that $I_1 \cup I_2 = I^+$ as $\beta_i \notin C^-$ because by assumption $\beta_i(d) > 0$ for all $i \in I^+$. By root finiteness, I_2 is a finite set.

Now let M denote the set of all minimal weights (in the root order defined in §1.3) of the finite set $\{\sum_{i \in J_2} \beta_i : J_2 \subset I_2\}$. Thus, M is finite. Define the set of weights in \mathcal{P} :

$$T := \{\Lambda - p_j - \gamma : j = 1 \dots m, \gamma \in M\},$$

which is a finite set and p_j 's are coset representatives defined earlier. We will show that elements of T "cover" all weights of $\text{Spin}(V)$ in the root order.

Due to the character formula above, any weight of $\text{Spin}(V)$ is of the form $\Lambda - \sum_{i \in J} \beta_i = \Lambda - \sum_{i \in J_1} \beta_i - \sum_{i \in J_2} \beta_i$, for some set $J \subset I^+$ and $J_1 := J \cap I_1$ and $J_2 := J \cap I_2$. Let λ be any such weight of $\text{Spin}(V)$. For the sum $\sum_{i \in J_1} \beta_i$ in λ , each β_i can be written as $p_j + b_j$ for some j and $b_j \in \mathcal{P}_R \cap C^+$. Now any finite sum of coset representatives p_j 's can be written as $p_k + a$ for $a \in \mathcal{P}_R \cap C^+$. So $\sum_{i \in J_1} \beta_i = p_k + a + \sum_{j=1}^m b_j = p_k + b$ for $b \in \mathcal{P}_R \cap C^+$. Write $c := \sum_{i \in J_2} \beta_i$. Therefore, $\lambda = \Lambda - (p_k + b) - c$. Choose an element $\gamma \in M$ such that $\gamma \leq c$ so that $c - \gamma \in \mathcal{P}_R \cap C^+$. Set $t := \Lambda - p_k - \gamma \in T$. Then $t - \lambda = b + c - \gamma \in \mathcal{P}_R \cap C^+$ so that $\lambda \leq t$. This proves that $\text{Spin}(V) \in \mathcal{O}$.

(\Rightarrow)

Suppose V is not root finite. Refer to the character formula given at the beginning of the Proof of the Proposition 1. Since V is not root finite, there exists an infinite set J such that $\beta_i \in \mathcal{P} - C$ for all $i \in J$ (See definitions of \mathcal{P} and C in Proof of

Proposition 1(5)). Extend the collection of simple positive roots, $\{\alpha_k\}_{k=1}^n$, of \mathfrak{g} to a basis of \mathfrak{h}^* , say:

$$\{\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_P\}.$$

For $i \in J$, let

$$\beta_i = \sum_{k=1}^n c_{i,k} \alpha_k = \beta_{i,1} + \beta_{i,2}$$

where $\beta_{i,1}$ lies in the real space spanned by $\{\alpha_1, \dots, \alpha_n\}$ and $\beta_{i,2}$ lies in the real space spanned by $\{\alpha_{n+1}, \dots, \alpha_P\}$. Because $\beta_i \in \mathcal{P} - C$, for each $i \in J$, either $\beta_{i,2} \neq 0$ or $\beta_{i,2} = 0$ and there exists a k such that $1 \leq k \leq n$ for which $c_{i,k} < 0$.

Thus, we may find an infinite set $J_1 \subset J$ and a k such that if $n+1 \leq k \leq P$ then $c_{i,k}$ is of same sign for all $i \in J_1$ and if $1 \leq k \leq n$ then $c_{i,k} < 0$ for all $i \in J_1$.

Define $\gamma_K := \sum_{i \in K} \beta_i$ for K a finite subset of J_1 . Also, let $\lambda_K := \Lambda - \gamma_K$. Then λ_K is a weight of $\text{Spin}(V)$ for each $K \subset J_1$. If $n+1 \leq k \leq P$ then λ_K when expressed in terms of above basis, will have coefficient of α_k unbounded (above or below) when K varies over finite subsets of J_1 . If $1 \leq k \leq n$ then this coefficient will be unbounded above. Thus these weights of $\text{Spin}(V)$ can not be bounded above in root order by finitely many weights from the weight lattice. Thus $\text{Spin}(V) \notin \mathcal{O}$. \square

Proof of Proposition 1(7). $V(\rho) \cong \text{Spin}_0(\mathfrak{g})$.

For the distinguished element d of \mathfrak{g} , the character formula given at the beginning of the proof of Proposition 1 leads to:

$$\text{Char Spin}_0(\mathfrak{g}) = e^\Lambda \prod_{\alpha \in R^+} (1 + e^{-\alpha}),$$

where the positive roots R^+ correspond to the d -positive weights $\{\alpha \in R : \alpha(d) > 0\}$ of adjoint representation \mathfrak{g} . Also, $\Lambda = \sum_{i=1}^n c_i \Lambda_i$, for Λ_i the i th fundamental weight of \mathfrak{g} . The formula above is the character formula for $V(\rho)$ if $c_i = 1$.

Restrict the adjoint representation to the 4-dimensional subalgebra $\mathfrak{d}_i := \mathfrak{s}_i \oplus \mathbb{C}d \subset \mathfrak{g}$, where $\mathfrak{s}_i \cong \mathfrak{sl}_2(\mathbb{C})$ corresponds to the simple root α_i and decompose the adjoint representation into finite dimensional \mathfrak{d}_i -orthogonal-irreducibles: $\mathfrak{g} \downarrow \cong \bigoplus_k V_k$. Here \downarrow denotes the restriction to \mathfrak{d}_i . Thus, by Proposition 1(4):

$$\text{Spin}_0(\mathfrak{g} \downarrow) \cong \text{Spin}_0(V_1) \otimes \text{Spin}_0(V_2) \otimes \dots$$

Using the character formula for Spin_0 for finite-dimensional representations, [P], the distinguished weight λ_k of $\text{Spin}_0(V_k)$ is the half sum of d -positive weights of V_k . Consider a highest weight vector $v = \sum_{\beta} a_{\beta} X_{\beta}$ of V_k for β 's positive roots of \mathfrak{g} . If $\beta = \sum_{\alpha} b_{\alpha} \alpha$ for α 's simple roots of \mathfrak{g} , then $\text{ad}(X_{-\alpha_i})^l(X_{\beta})$ is a positive root vector or zero for all l unless $\beta = \alpha_i$. This leads to $\lambda_k(\alpha_i^{\vee}) = 0$ if $v \notin \mathfrak{g}^{(\alpha_i)}$ and $\lambda_k(\alpha_i^{\vee}) = 1$ if $v \in \mathfrak{g}^{(\alpha_i)}$. Thus, $c_i = \sum_k \lambda_k(\alpha_i^{\vee}) = 1$ for all i . \square

3.2.2. Proof of Proposition 2. Refer to the character formula given before the proof of Proposition 1 which leads to:

$$\text{Char Spin}_0(\mathfrak{g}) = e^\Lambda \prod_{\beta(d) > 0} (1 + e^{-\beta}),$$

where we can write, $\Lambda = \sum_{i=1}^n c_i \Lambda_i$, for Λ_i the i th fundamental weight of \mathfrak{g} . Lets call the weights β of V such that $\beta(d) > 0$ as d -positive weights of V . If V is finite dimensional then by [P], $\Lambda = \sum_{\beta(d) > 0} \frac{1}{2} m_{\beta} \beta$. When V is infinite dimensional, this is an infinite sum but as in proof of Proposition 1(7), we may restrict V to $\mathfrak{d}_i := \mathfrak{s}_i \oplus \mathbb{C}d \subset \mathfrak{g}$ and decompose $V \downarrow \cong \bigoplus_k V_k$ into finite dimensional \mathfrak{d}_i -orthogonal-irreducibles V_k . Then the distinguished weight λ_k of $\text{Spin}_0(V_k)$ is the half sum

of d -positive weights of V_k . But for a weight β of V if both β and $s_i(\beta)$ are d -positive, that is, $\beta(d) > 0$ and $s_i\beta(d) > 0$ they will not contribute to $\lambda_k(\alpha_i^\vee)$ because $\beta(\alpha_i^\vee) + s_i\beta(\alpha_i^\vee) = 0$. Finally, by Proposition 1(4), $c_i = \sum_k \lambda_k(\alpha_i^\vee)$, and the result follows.

3.3. Proof of Theorem 1. Proposition 1(7), says:

$$\mathrm{Spin}_0(\tilde{\mathfrak{g}}) \cong V(\tilde{\rho})$$

We restrict the adjoint representation $\tilde{\mathfrak{g}}$ to \mathfrak{g} and apply Spin_0 with respect to \mathfrak{g} . Using, Proposition 1(4), we get the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\mathrm{Spin}_0} & V(\tilde{\rho}) \\ \downarrow & & \downarrow \\ \mathfrak{g} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots & \xrightarrow{\mathrm{Spin}_0} & V(\rho) \otimes W_1 \otimes W_2 \otimes \cdots \end{array}$$

where vertical arrows denote restriction. The diagram commutes for the following reason. Fix a d -finite, orthogonal $\tilde{\mathfrak{g}}$ -representations V (such as adjoint representation of $\tilde{\mathfrak{g}}$). Clearly, Spin commutes with restriction, $\downarrow_{\tilde{\mathfrak{g}}}$ when acted on V . Express, Spin in terms of Spin_0 using Proposition 1(2). Since $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is a d -embedding and V is a d -finite representation, the non-zero $\tilde{\mathfrak{g}}$ -weights of V restrict to non-zero \mathfrak{g} -weights. Thus the dimension of the zero weight space does not change upon restriction. Hence Spin_0 also commutes with $\downarrow_{\tilde{\mathfrak{g}}}$ when applied to V .

3.4. Proof of Theorem 2. Let $\chi_\lambda := \mathfrak{Char} V(\lambda)$. For any $\chi = \sum_{\lambda \in I} c_\lambda e^\lambda$, $I \subset \mathcal{P}$ the weight lattice, define $\chi^{(2)} = \sum_{\lambda \in I} c_\lambda e^{2\lambda}$. Weyl character formula for character of an irreducible \mathfrak{g} -representation with highest weight λ , says:

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho},$$

where the skew-symmetrizer, $A_\mu := \sum_{w \in \mathcal{W}} \mathrm{sign}(w) e^{w(\mu)}$ and \mathcal{W} is Weyl group.

$$\begin{aligned} \chi_{2\tilde{\mu}+\tilde{\rho}} &= \frac{A_{2(\tilde{\mu}+\tilde{\rho})}}{A_{\tilde{\rho}}} \\ &= \frac{A_{2(\tilde{\mu}+\tilde{\rho})}}{A_{2\tilde{\rho}}} \frac{A_{2\tilde{\rho}}}{A_{\tilde{\rho}}} \\ \chi_{2\tilde{\mu}+\tilde{\rho}} &= \frac{A_{\tilde{\mu}+\tilde{\rho}}^{(2)}}{A_{\tilde{\rho}}^{(2)}} \frac{A_{2\tilde{\rho}}}{A_{\tilde{\rho}}} \end{aligned}$$

$$(3) \quad \text{Thus, } \chi_{2\tilde{\mu}+\tilde{\rho}} = \chi_{\tilde{\mu}}^{(2)} \chi_{\tilde{\rho}}$$

Let $\chi_{\tilde{\mu}} \downarrow = \sum_i \chi_{\mu_i}$, where \downarrow denotes restriction from $\tilde{\mathfrak{g}}$ to \mathfrak{g} . Also, let $W := W_1 \otimes W_2 \otimes \cdots$, where W_k 's are defined in Theorem 1.

$$\begin{aligned} \chi_{2\tilde{\mu}+\tilde{\rho}} \downarrow &= \chi_{\tilde{\mu}}^{(2)} \downarrow \chi_{\tilde{\rho}} \downarrow \\ &= \left(\sum_{i=1}^k \chi_{\mu_i}^{(2)} \right) (\chi_\rho \mathfrak{Char}(W)) \quad \text{by Theorem 1.} \\ &= \left(\sum_{i=1}^k \chi_{\mu_i}^{(2)} \chi_\rho \right) \mathfrak{Char}(W) \end{aligned}$$

$$\text{Therefore, } \chi_{2\tilde{\mu}+\tilde{\rho}} \downarrow = \left(\sum_{i=1}^k \chi_{2\mu_i+\rho} \right) \mathfrak{Char}(W),$$

where the last equality is due to the same reason as for equation (3). This proves Theorem 2. \square

3.5. Proof of Propositions 3-4.

3.5.1. *Proof of Proposition 3.* We will use $S_\lambda(x_1, x_2, \dots, x_n)$ to denote irreducible characters for $\tilde{\mathfrak{g}} = \mathfrak{sl}_n \mathbb{C}$ and χ_k for irreducible character of principal $\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}$ with highest weight $k\varpi$. The restriction in this case corresponds to setting x_i , in S_λ as $q^{(n+1-2i)/2}$ for $q = e^\alpha$, and α the root of \mathfrak{g} . We fix the following notation:

$$S_\lambda \downarrow := S_\lambda(q^{(n-1)/2}, q^{(n-3)/2}, \dots, q^{-(n-1)/2}),$$

and

$$S_\lambda^{(2)} := S_\lambda(x_1^2, x_2^2, \dots, x_n^2).$$

Then character of adjoint representation of $\tilde{\mathfrak{g}}$ when restricted to \mathfrak{g} gives:

$$S_\theta \downarrow = \chi_2 + \chi_4 + \dots + \chi_{2(n-1)},$$

where θ is the highest root of $\tilde{\mathfrak{g}}$. We apply Spin_0 on corresponding representations, and use the character formula for Spin_0 from Proposition 2. Then using Theorem 1 we obtain:

$$S_\rho \downarrow = \chi_1 \cdot u_1 \cdot u_2 \cdots u_{n-2}$$

where $u_k := \prod_{j=1}^{k+1} (q^{j/2} + q^{-j/2})$ and $\rho = (n-1, n-2, \dots, 1, 0)$. Now, Theorem 2, leads to:

$$S_{2\mu+\rho} \downarrow = (S_\mu^{(2)} \downarrow \chi_1) \cdot u_1 \cdot u_2 \cdots u_{n-2}$$

where all $n-1$ factors on right hand side are characters of $\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}$. In order to translate this in language of principal specialization, we observe the following identity:

$$S_\lambda(1, q, q^2, \dots, q^{n-1}) = q^N S_\lambda \downarrow$$

where $N = \frac{n-1}{2} \sum_{i=1}^n \lambda_i$, for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Using the above formula, we obtain:

$$\begin{aligned} S_{2\mu+\rho}(1, q, q^2, \dots, q^{n-1}) \\ = \left(q^{\binom{n}{3}} (1+q) S_\mu(1, q^2, q^4, \dots, q^{2n-2}) \right) \cdot w_1(q) \cdot w_2(q) \cdots w_{n-2}(q), \end{aligned}$$

where $w_k(q) = (1+q)(1+q^2) \cdots (1+q^{k+1})$, where all $(n-1)$ factors on the right are symmetric unimodal as they have been obtained from $\mathfrak{sl}_2 \mathbb{C}$ -characters.

3.5.2. *Proof of Proposition 4.* We list all graph automorphisms of Dynkin diagrams, \tilde{D} , of simple Lie algebras $\tilde{\mathfrak{g}}$ and the corresponding fixed subalgebras \mathfrak{g} . We obtain the factorizations using Theorem 1. Let θ and $\tilde{\theta}$ be the highest roots of \mathfrak{g} and $\tilde{\mathfrak{g}}$ respectively so that $V(\theta)$ and $V(\tilde{\theta})$ denote their adjoint representations. Let $V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \dots$, so that $V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes W_1 \otimes W_2 \otimes \dots$. We specify \mathfrak{p}_i 's and W_i 's. The fact that $\text{Spin}_0(\mathfrak{p}_i) \cong W_i$ will be proved in Proposition 9.

(1) Graph automorphisms of order 2.

(a) $\tilde{\mathfrak{g}} = A_{2n-1} = \mathfrak{sl}_{2n} \mathbb{C}$.

$\mathfrak{g} = C_n = \mathfrak{sp}_{2n} \mathbb{C}$.

$i \xleftrightarrow{\phi} 2n-i$.

$V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus V(\theta_s)$, where θ_s is the highest short root of \mathfrak{g} .

$V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes V(\rho_s)$.

- (b) $\tilde{\mathfrak{g}} = D_{n+1} = \mathfrak{so}_{2n+2}\mathbb{C}$.
 $\mathfrak{g} = B_n = \mathfrak{so}_{2n+1}\mathbb{C}$.
 $n \xleftrightarrow{\phi} n+1$, interchanges two forked nodes and fixes others.
 $V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus V(\theta_s)$, where θ_s is the highest short root of \mathfrak{g} .
 $V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes V(\rho_s)$.
- (c) $\tilde{\mathfrak{g}} = E_6$.
 $\mathfrak{g} = F_4$.
 $1 \xleftrightarrow{\phi} 5 \quad 2 \xleftrightarrow{\phi} 4$ and fixes others. (See page 53 [K] for ordering of nodes)
 $V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus V(\theta_s)$, where θ_s is the highest short root of \mathfrak{g} .
 $V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes V(\rho_s)$.
- (d) $\tilde{\mathfrak{g}} = A_{2n} = \mathfrak{sl}_{2n+1}\mathbb{C}$.
 $\mathfrak{g} = B_n = \mathfrak{so}_{2n+1}\mathbb{C}$.
 $i \xleftrightarrow{\phi} 2n+1-i$.
 $V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus V(2\theta_s)$, where θ_s is the highest short root of \mathfrak{g} .
 $V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes V(\rho + 2\rho_s)$.
- (2) Graph automorphism of order 3.
- (a) $\tilde{\mathfrak{g}} = D_4$.
 $\mathfrak{g} = G_2$.
 ϕ cyclically permutes the three outer nodes and fixes the middle node.
 $V(\tilde{\theta}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\theta) \oplus V(\theta_s) \oplus V(\theta_s)$, where θ_s is the highest short root of \mathfrak{g} .
 $V(\tilde{\rho}) \downarrow_{\tilde{\mathfrak{g}}}^{\tilde{\mathfrak{g}}} \cong V(\rho) \otimes (V(\rho_s) + V(0)) \otimes (V(\rho_s) + V(0))$.

3.6. Proof of Propositions 5-8.

3.6.1. *Proof of Proposition 5.* We will prove Proposition 5 using Proposition 2. Recall that $-N < \beta(d_1), \theta(d_1) < N$ for all weights $\beta \in T$ of V . Now all weights of \hat{V} are of the form $k\delta + \beta$ with multiplicity m_β for $k \in \mathbb{Z}$ and β a weight of V . Thus, for $\hat{d} = Nd + d_1$, $(k\delta + \beta)(\hat{d}) = kN + \beta(d_1) > 0$ if and only if $k > 0$ or $k = 0$, $\beta(d_1) > 0$ by the definition of N . Proposition 2 leads to:

$$\mathfrak{Char} \text{Spin}_0(\hat{V}) = e^\Lambda \prod_{\beta(d_1) > 0} (1 + e^{-\beta})^{m_\beta} \prod_{\substack{k > 0 \\ \beta \in T}} (1 + e^{-\beta - k\delta})^{m_\beta},$$

where $\Lambda = \sum_{i=0}^n c_i \Lambda_i$, c_i as defined in Proposition 2. It is easy to verify that for $i = 1 \dots n$, $c_i = \sum \frac{1}{2} m_\beta \beta(\alpha_i^\vee)$ summing over weights β of V (as opposed to \hat{V}) such that $\beta(d_1) > 0$ and $s_i(\beta)(d_1) < 0$ because $s_i(k\delta + \beta) = k\delta + s_i(\beta)$. Further, since V is finite dimensional, we may drop the condition $s_i(\beta)(d_1) < 0$ and sum over all weights β of V such that $\beta(d_1) > 0$ because $\beta(\alpha_i^\vee) + s_i(\beta)(\alpha_i^\vee) = 0$. Thus:

$$c_i = \sum_{\beta(d_1) > 0} \frac{1}{2} m_\beta \beta(\alpha_i^\vee), \quad i = 1, \dots, n$$

For $i = 0$ case, $(k\delta + \beta)(\alpha_0^\vee) = -\beta(\theta^\vee)$ as $\alpha_0 = K - \theta^\vee$. Also, $m_{k\delta \pm \beta} = m_\beta$. Therefore by replacing β by $-\beta$, we get $c_0 = \sum \frac{1}{2} m_\beta \beta(\theta^\vee)$, summing over all $k \in \mathbb{Z}$ and $\beta \in T$ such that $(k\delta + \beta)(\hat{d}) > 0$ and $s_0(k\delta + \beta)(\hat{d}) < 0$ which simplifies to the inequality:

$$(4) \quad \frac{\beta(d_1)}{N} < k < \beta(\theta^\vee) + \frac{s_\theta \beta(d_1)}{N}$$

where s_θ denotes the reflection corresponding to the highest root θ of \mathfrak{g} .

Now, by definition of N , $\frac{\beta(d_1)}{N}$ and $\frac{s_\theta \beta(d_1)}{N}$ are fractions. So the inequality (4) implies that $\beta(\theta^\vee) \geq 0$. Since c_0 involves summing $\frac{1}{2}m_\beta \beta(\theta^\vee)$ over inequality (4), we may sum over $\beta(\theta^\vee) > 0$. Consider the following cases:

- Case 1 : $\beta(\theta^\vee) > 0$, $\beta(d_1) < 0$ ($\Rightarrow s_\theta \beta(d_1) < 0$).
- Case 2 : $\beta(\theta^\vee) > 0$, $\beta(d_1) > 0$ and $s_\theta \beta(d_1) > 0$.
- Case 3 : $\beta(\theta^\vee) > 0$, $\beta(d_1) > 0$ and $s_\theta \beta(d_1) < 0$.

In Case 1, inequality (4) $\Rightarrow 0 \leq k \leq \beta(\theta^\vee) - 1$. In Case 2, inequality (4) $\Rightarrow 1 \leq k \leq \beta(\theta^\vee)$ and in Case 3, it implies $1 \leq k \leq \beta(\theta^\vee) - 1$. Thus, we get:

$$\begin{aligned} c_0 &= \sum_{\text{Case 1}} \frac{1}{2} m_\beta \beta(\theta^\vee)^2 + \sum_{\text{Case 2}} \frac{1}{2} m_\beta \beta(\theta^\vee)^2 + \sum_{\text{Case 3}} \frac{1}{2} m_\beta (\beta(\theta^\vee)^2 - \beta(\theta^\vee)) \\ &= \sum_{\text{Case 1,2,3}} \frac{1}{2} m_\beta \beta(\theta^\vee)^2 - \sum_{\text{Case 3}} \frac{1}{2} m_\beta \beta(\theta^\vee). \end{aligned}$$

Now, in the first sum the union of the three cases leads to the case $\beta(\theta^\vee) > 0$ and due to the square in the sum, it is equivalent to summing over $\beta(d_1) > 0$. In the second sum over Case 3, we may drop $(\beta(\theta^\vee) > 0)$ as it is implied by $\beta(d_1) > 0$ and $s_\theta \beta(d_1) < 0$. Further, we may also drop $(s_\theta \beta(d_1) < 0)$ for the same reason which led to the expression for c_i , $i = 1 \cdots n$. Therefore:

$$\begin{aligned} c_0 &= \sum_{\beta(d_1) > 0} \frac{1}{2} m_\beta \beta(\theta^\vee)^2 - \sum_{\beta(d_1) > 0} \frac{1}{2} m_\beta \beta(\theta^\vee), \\ c_i &= \sum_{\beta(d_1) > 0} \frac{1}{2} m_\beta \beta(\alpha_i^\vee) \quad i = 1, 2 \cdots n, \end{aligned}$$

and $\Lambda = \sum_{i=0}^n c_i \Lambda_i$ which leads to Proposition 5 using $\Lambda_i = \varpi_i + a_i^\vee \Lambda_0$ and $\theta^\vee = \sum_{i=1}^n a_i^\vee \alpha_i^\vee$. Here, ϖ_i is the i th-fundamental weight of \mathfrak{g} .

3.6.2. Proof of Proposition 6. Exactly same as that of Theorem 1 using property of Spin_0 given in Proposition 1(4).

3.6.3. Proof of Proposition 7. Direct consequence of Propositions 6 and 1(7).

3.6.4. Proof of Proposition 8. This is just Theorem 2 for affine Lie algebras where we make use of Proposition 7.

3.7. Proof of Propositions 9-10. We will use R, R_s and R_l to denote the set of all roots, short roots (if any) and long roots (if any) of a finite-dimensional semi-simple Lie algebra \mathfrak{g} with distinguished element $d = \rho^\vee$, the sum of all fundamental co-weights of \mathfrak{g} . Similarly R^+, R_s^+ and R_l^+ will denote the set of all positive roots, positive short roots and positive long roots.

3.7.1. Proof of Proposition 9. This classification was done by [P]. Here we give proofs of the facts about $\text{Spin}_0(V)$ for each of the 3 cases.

Proof for Case 1. It is given in Proposition 1(7) earlier which uses Weyl denominator identity. We use it again to prove other cases also. \square

Proof for Case 2. Let $\chi := \text{Char Spin}_0 V(\theta_s)$. By Proposition 2:

$$\chi = e^{\rho_s} \prod_{\alpha \in R_s^+} (1 + e^{-\alpha}).$$

Weyl denominator identity is:

$$\begin{aligned}
 A_\rho &= e^\rho \prod_{\alpha \in R_s^+} (1 - e^{-\alpha}) \prod_{\alpha \in R_l^+} (1 - e^{-\alpha}). \\
 \chi A_\rho &= e^{\rho_s + \rho} \prod_{\alpha \in R_s^+} (1 - e^{-2\alpha}) \prod_{\alpha \in R_l^+} (1 - e^{-\alpha}) \\
 (5) \quad &= e^{\rho_s + \rho} \prod_{\alpha \in (2R_s^+ \cup R_l^+)} (1 - e^{-\alpha}).
 \end{aligned}$$

If for \mathfrak{g} , $(\|\theta\|^2 / \|\theta_s\|^2) = 2$ (as in Case 2) then $(2R_s \cup R_l)$ forms the root system of the dual algebra, denoted by $\tilde{\mathfrak{g}}$, of \mathfrak{g} . Then the half sum of positive roots $\tilde{\rho}$ of $\tilde{\mathfrak{g}}$ is given by $\tilde{\rho} = \rho_s + \rho$. Thus by Weyl denominator identity for $\tilde{\mathfrak{g}}$ and equation (5):

$$\chi A_\rho = \tilde{A}_{\tilde{\rho}} = \tilde{A}_{\rho_s + \rho}.$$

Since, \mathfrak{g} and $\tilde{\mathfrak{g}}$ have the same Weyl group and A_μ is the anti-symmetrizer of μ w.r.t. Weyl group, $\tilde{A}_{\rho_s + \rho} = A_{\rho_s + \rho}$.

$$\begin{aligned}
 \Rightarrow \chi A_\rho &= A_{\rho_s + \rho}. \\
 \Rightarrow \chi &= \frac{A_{\rho_s + \rho}}{A_\rho} = \mathfrak{Char} V(\rho_s).
 \end{aligned}$$

□

Proof for Case 3. For $n \geq 2$, Panyushev [P, Prop. 3.8] showed that the set of all nonzero weights of $V(2\theta_s)$ is $S = 2R_s \cup R_s \cup R_l$ with multiplicity of each nonzero weight as 1. For $n = 1$, define $R_s := R$, $R_s^+ := R^+$ and $R_l := \{\}$, $R_l^+ := \{\}$. Also define $\prod_{\alpha \in \{\}} f(\alpha) := 1$ for any function f . Let $\chi := \mathfrak{Char} \text{Spin}_0 V(2\theta_s)$. Thus by Proposition 2:

$$\begin{aligned}
 \chi &= e^{2\rho_s + \rho} \prod_{\alpha \in R_s^+} (1 + e^{-2\alpha}) \prod_{\alpha \in R_s^+} (1 + e^{-\alpha}) \prod_{\alpha \in R_l^+} (1 + e^{-\alpha}), \\
 A_\rho &= e^\rho \prod_{\alpha \in R_s^+} (1 - e^{-\alpha}) \prod_{\alpha \in R_l^+} (1 - e^{-\alpha}).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \chi A_\rho &= e^{2\rho_s + 2\rho} \prod_{\alpha \in R_s^+} (1 - e^{-4\alpha}) \prod_{\alpha \in R_l^+} (1 - e^{-2\alpha}) \\
 &= A_{4\rho} = A_{2(\rho_s + \rho)} \text{ as } \rho_s = \rho \text{ for } n = 1 \\
 &= e^{2(\rho_s + \rho)} \prod_{\alpha \in (2R_s^+ \cup R_l^+)} (1 - e^{-2\alpha}) \text{ (For } n \geq 2) \\
 &= \tilde{A}_{2\tilde{\rho}} \text{ (By comparing with } \tilde{A}_{\tilde{\rho}} \text{ of dual algebra } \tilde{\mathfrak{g}} \text{ for } n \geq 2) \\
 &= A_{2(\rho_s + \rho)} \text{ (As } \tilde{\rho} = \rho_s + \rho \text{ as in case 2)} \\
 \Rightarrow \chi &= \frac{A_{2(\rho_s + \rho) + \rho}}{A_\rho} = \mathfrak{Char} V(2\rho_s + \rho) \text{ (For } n \geq 1).
 \end{aligned}$$

□

3.7.2. Proof of Proposition 10.

Proof of (\Rightarrow). Let $\text{Spin}_0(\hat{V})$ be irreducible and denote its character by χ . Also let S denote the set of all non-zero weights of V and $q := e^\delta$. Then by Proposition 5:

$$\chi = \mathfrak{Char} \text{Spin}_0(V) e^{c\Lambda_0} \prod_{k>0, \beta \in S} (1 + e^{-\beta} q^{-k})^{m_\beta} \prod_{k>0} (1 + q^{-k})^{m_0}$$

where $c = \sum_{\beta \in S^+} \frac{1}{2} m_\beta \beta (\theta^\vee)^2$. Suppose that $\mathfrak{Char} \text{Spin}_0(V) = \sum_{i=1}^s \chi_{\nu_i}$ where χ_{ν_i} is the irreducible character with highest weight ν_i . We first show that $s = 1$ meaning V is coprimary. Weyl denominator identity for $\hat{\mathfrak{g}}$ is:

$$\hat{A}_{\hat{\rho}} = e^{\rho + h^\vee \Lambda_0} \prod_{\alpha \in R^+} (1 - e^{-\alpha}) \prod_{k>0, \alpha \in R} (1 - e^{-\alpha - k\delta}) \prod_{k>0} (1 - e^{-k\delta})^n$$

where h^\vee is the dual Coxeter number. Using Weyl denominator identity for \mathfrak{g} , namely, $A_\rho = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})$ and writing $q = e^\delta$, we can say:

$$\hat{A}_{\hat{\rho}} = A_\rho e^{h^\vee \Lambda_0} \prod_{k>0, \alpha \in R} (1 - e^{-\alpha} q^{-k}) \prod_{k>0} (1 - q^{-k})^n$$

Multiplying χ with $\hat{A}_{\hat{\rho}}$, we get:

$$\begin{aligned} \chi \cdot \hat{A}_{\hat{\rho}} &= \left(\sum_{i=1}^s \chi_{\nu_i} A_\rho \right) e^{(c+h^\vee)\Lambda_0} + (\dots)q^{-1} + (\dots)q^{-2} + \dots \\ &= \left(\sum_{i=1}^s A_{\nu_i + \rho} \right) e^{(c+h^\vee)\Lambda_0} + (\dots)q^{-1} + (\dots)q^{-2} + \dots \end{aligned}$$

$\chi \cdot \hat{A}_{\hat{\rho}}$ will contain the term $e^{\nu_i + \rho + (c+h^\vee)\Lambda_0} = e^{\nu_i + c\Lambda_0 + \hat{\rho}}$ for each $i = 1 \dots s$ where ν_i is a dominant weight of \mathfrak{g} . By character of $\text{Spin}_0(V)$, all its weights are of the form: $\frac{1}{2} \sum_{\beta \in S^+} a_\beta \beta$ for some $-m_\beta \leq a_\beta \leq m_\beta$. Since $c = \frac{1}{2} \sum_{\beta \in S^+} m_\beta \beta (\theta^\vee)^2$, $\nu_i(\theta^\vee) \leq c$. Also, since ν_i is a dominant weight of \mathfrak{g} , this shows that $\nu_i + c\Lambda_0$ is a dominant weight of $\hat{\mathfrak{g}}$ for all $i = 1 \dots s$. Hence χ contains irreducible $\chi_{\nu_i + c\Lambda_0}$ for each i in its decomposition into irreducibles. So, s must be 1 because $\chi = \mathfrak{Char} \text{Spin}_0(\hat{V})$ is irreducible.

Next, we show that when $V = V(2\theta_s)$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$ then $\text{Spin}_0(\hat{V})$ is not irreducible. By [P, Prop 3.8], the set of all nonzero weights of $V(2\theta_s)$ is $S = 2R_s \cup R_s \cup R_l$ with multiplicity of each nonzero weight as 1. This holds for $n = 1$ also, if we define $R_s := R$, $R_l := \{\}$ and $R_s^+ := R^+$, $R_l^+ := \{\}$. Let $\prod_{\alpha \in \{\}} f(\alpha) := 1$ for any function f and $S^+ = 2R_s^+ \cup R_s^+ \cup R_l^+$. Now by Proposition 5:

$$\mathfrak{Char} \text{Spin}_0(\hat{V}) = e^{2\rho_s + \rho + c\Lambda_0} \prod_{\beta \in S^+} (1 + e^{-\beta}) \prod_{k>0, \beta \in S} (1 + e^{-\beta - k\delta}) \prod_{k>0} (1 + e^{-k\delta})^{m_0}.$$

We calculate the level c of the representation $\text{Spin}_0(\hat{V})$ as follows: For $n \geq 2$, $R_s^+ = \{L_i\}$, $R_l^+ = \{L_i \pm L_j : i < j\}$, and the dual positive roots are $\{2H_i\} \cup \{H_i \pm H_j : i < j\}$, where $\{H_i : 1 \leq i \leq n\}$ is the dual basis of $\{L_i : 1 \leq i \leq n\}$. Then $\theta = L_1 + L_2$ and $\theta^\vee = H_1 + H_2 = H_{\alpha_1} + H_{\alpha_n} + 2 \sum_{i=2}^{n-1} H_{\alpha_i}$. Thus:

$$c = \begin{cases} 2n + 3 & \text{for } n \geq 2 \\ 10 & \text{for } n = 1. \end{cases}$$

Also, the multiplicity of zero weight space, $m_0 = n$, the rank of \mathfrak{g} , by Panyushev [P, Prop 3.8]. Thus, $\mathfrak{Char} \text{Spin}_0(\hat{V})$ reduces to:

$$\chi := \mathfrak{Char} \text{Spin}_0(\hat{V}) = e^{2\rho_s + \rho} e^{c\Lambda_0} \prod_{\beta \in S^+} (1 + e^{-\beta}) \prod_{k>0, \beta \in S} (1 + e^{-\beta - k\delta}) \prod_{k>0} (1 + e^{-k\delta})^n.$$

The highest dominant weight appearing in χ is $\Lambda := 2\rho_s + \rho + c\Lambda_0$. To prove that χ is not an irreducible character of $\widehat{\mathfrak{g}}$, it is enough to produce another dominant weight appearing in χ , say, Λ' such that $(\Lambda - \Lambda')$ can not be expressed as non-negative integral linear combination of simple positive roots of $\widehat{\mathfrak{g}}$. For $n \geq 2$, take $\Lambda' = 2\rho_s + \rho + 2L_1 - \delta + c\Lambda_0$ which corresponds to the term in χ , $e^{2\rho_s + \rho} e^{c\Lambda_0} e^{-\beta - k\delta}$ for $\beta = -2L_1 \in S$ and $k = 1$. Clearly, Λ' is dominant as $\lambda' := 2\rho_s + \rho + 2L_1$ is dominant weight of \mathfrak{g} and $\lambda'(\theta^\vee) = 2n + 2 \leq c = 2n + 3$. $\Lambda - \Lambda' = -2L_1 + \delta = \delta - (L_1 + L_2) - (L_1 - L_2) = \alpha_0 - \alpha_1$. For $n = 1$, take $\Lambda' = 2\rho_s + \rho + (2\alpha - \delta) + c\Lambda_0 = 7\rho - \delta + 10\Lambda_0$. Then $\lambda'(\theta^\vee) = 7 \leq 10$. Here, $\Lambda = 3\rho + 10\Lambda_0$. So, $\Lambda - \Lambda' = -4\rho + \delta = -2\alpha + \delta = (\delta - \alpha) - \alpha = \alpha_0 - \alpha_1$. \square

Proof of (\Leftarrow). Using Proposition 1(7), it is enough to prove :

$$V = V(\theta_s) \Rightarrow \text{Spin}_0(\hat{V}) = V(\hat{\rho}_s),$$

where, $\hat{\rho}_s := \rho_s + h_s^\vee \Lambda_0$, $h_s^\vee := \sum_i a_i^\vee$, i 's corresponding to short simple roots of $\widehat{\mathfrak{g}}$.

In the proof of Case 2 of Proposition 9, we dealt with finite dimensional Lie algebras \mathfrak{g} and its dual algebra $\widehat{\mathfrak{g}}$. In the same spirit, here we will deal with the affine Lie algebra $\widehat{\mathfrak{g}}$ for $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{f}_4\}$ and its Langlands dual (obtained by reversing the arrows of the Dynkin diagram of $\widehat{\mathfrak{g}}$) denoted by $\check{\mathfrak{g}}$. So if R denotes an object associated to \mathfrak{g} (say R = the set of roots of \mathfrak{g}) then the corresponding object (the (multi-)set of roots) associated to $\widehat{\mathfrak{g}}$, $\widehat{\mathfrak{g}}$ or $\check{\mathfrak{g}}$ will be denoted by \hat{R} , \check{R} and \check{R} respectively. Let $\chi = \mathcal{CH}\text{ar Spin}_0(\hat{V})$. In order to completely adapt the proof for Case 2 of Proposition 9, we will define multisets associated to roots of $\widehat{\mathfrak{g}}$, namely \hat{R}_s^+ and \hat{R}_l^+ and show the following Facts:

- (1) $\chi = e^{\hat{\rho}_s} \prod_{\alpha \in \hat{R}_s^+} (1 + e^{-\alpha})$.
- (2) $\hat{R}_s^+ \cup \hat{R}_l^+ = \hat{R}^+$ the multiset of all positive roots of $\widehat{\mathfrak{g}}$.
- (3) $2\hat{R}_s^+ \cup \hat{R}_l^+ = \check{R}^+$ the multiset of all positive roots of $\check{\mathfrak{g}}$.
- (4) $\check{\rho} = \hat{\rho}_s + \hat{\rho}$.

First we introduce the following notation: For any set A , define $A_{\{k\}} :=$ a multiset consisting of elements of A with each element appearing k times. Further, the multiset $A_{\{1\}}$ will be written as A .

Proof of Fact 2. Now, the multiset of all positive roots of $\widehat{\mathfrak{g}}$:

$$\hat{R}^+ := R^+ \cup \{\alpha + k\delta : \alpha \in R, k \in \mathbb{Z}_{>0}\} \cup \{k\delta : k \in \mathbb{Z}_{>0}\}_{n_s}$$

For $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{f}_4\}$, define:

$$\hat{R}_s^+ := R_s^+ \cup \{\alpha + k\delta : \alpha \in R_s, k \in \mathbb{Z}_{>0}\} \cup \{k\delta : k \in \mathbb{Z}_{>0}\}_{n_s}$$

where $n_s :=$ number of short simple positive roots of \mathfrak{g} . Similarly,

$$\hat{R}_l^+ := R_l^+ \cup \{\alpha + k\delta : \alpha \in R_l, k \in \mathbb{Z}_{>0}\} \cup \{k\delta : k \in \mathbb{Z}_{>0}\}_{n-n_s}.$$

Clearly, 2 is true. \square

Proof of Fact 3.

$$\begin{aligned} \mathfrak{g} \in \{\mathfrak{so}_{2n+1}\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{f}_4\} &\Rightarrow \left(\|\theta\|^2 / \|\theta_s\|^2 \right) = 2 \text{ in } \mathfrak{g} \\ \Rightarrow \widehat{\mathfrak{g}} \in \{B_n^{(1)}, C_n^{(1)}, F_4^{(1)}\} &\Rightarrow \check{\mathfrak{g}} \in \{A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}\}. \end{aligned}$$

Note that, $\check{\mathfrak{g}}$ is a *twisted* affine Lie algebra which contains the finite dimensional dual algebra $\widehat{\mathfrak{g}}$ of \mathfrak{g} . The set of all positive roots of $\check{\mathfrak{g}}$ is $2\hat{R}_s^+ \cup \hat{R}_l^+$. Now 3 is obvious for real roots or one can check directly using [K, Prop. 6.3]. We only need to check

the multiplicities of imaginary positive roots. We observe that, in $2\hat{R}_s^+ \cup \hat{R}_l^+$ the multiplicity of $2k\delta$ is $n_s + (n - n_s) = n$ and multiplicity of $(2k+1)\delta$ is $n - n_s$ which matches with multiplicities given by V. Kac [K, Corollary 8.3] for twisted affine Lie algebra $\check{\mathfrak{g}}$. \square

Proof of Fact 4. Let \prod_s, \prod_l denote the set of all simple positive short and long roots of $\hat{\mathfrak{g}}$ respectively. Also, let \sum_s, \sum_l denote the set of all simple positive coroots corresponding to short and long roots of $\hat{\mathfrak{g}}$ respectively. Then $(2\prod_s \cup \prod_l)$ and $(\frac{1}{2}\sum_s \cup \sum_l)$ forms the sets of simple positive roots and coroots of $\check{\mathfrak{g}}$ respectively. From this we can conclude that $\check{\rho} = \hat{\rho}_s + \hat{\rho}$. \square

Proof of Fact 1. Refer to Proposition 5 for expression for $\chi := \mathfrak{Char} \text{Spin}_0(\hat{V})$. It's easy to check that set of all nonzero weights of $V = V(\theta_s)$ is $S = R_s$. Since multiplicity of zero weight space is n_s , Proposition 5 leads to :

$$\chi = e^{\nu + c\Lambda_0} \prod_{\alpha \in \hat{R}_s^+} (1 + e^{-\alpha})$$

where, $\nu := \frac{1}{2} \sum_{\alpha \in R_s^+} \alpha = \rho_s$ and $c = \frac{1}{2} \sum_{\alpha \in R_s^+} \alpha(\theta^\vee)^2$.

We will show that $\nu + c\Lambda_0 = \hat{\rho}_s$. Since, θ is always a long root for $\mathfrak{g} \in \{\mathfrak{so}_{2n+1}\mathbb{C}, \mathfrak{sp}_{2n}\mathbb{C}, \mathfrak{f}_4\}$, $\alpha(\theta^\vee)$ is 0 or 1 for all $\alpha \in R_s^+$ due to following Lemma.

Lemma 3.

$$\alpha \in R^+ \setminus \{\theta\} \Rightarrow \alpha(\theta^\vee) = 0 \text{ or } 1.$$

Proof. Verify the following facts about any finite root system $R = R^+ \cup R^-$.

- (1) $\alpha \in R^+ \Rightarrow \alpha + \theta \notin R$.
- (2) $\alpha \in R^- \Rightarrow \alpha - \theta \notin R$.
- (3) For $\alpha \in R^+$, $\alpha - \theta \in R \Rightarrow \alpha - \theta \in R^- \Rightarrow \alpha - 2\theta \notin R$.

Consider the restriction of the adjoint representation, $V(\theta)$, to the $\mathfrak{sl}_2\mathbb{C}$ corresponding to θ , $\mathfrak{s}_\theta := \mathbb{C}X_\theta \oplus \mathbb{C}X_{-\theta} \oplus \mathbb{C}\theta^\vee$. Using above facts, we conclude; for a fixed $\alpha \in R^+ \setminus \{\theta\}$:

- If $\alpha - \theta \notin R$, then $\mathbb{C}X_\alpha$ is a trivial irreducible component of $V(\theta)$ as an \mathfrak{s}_θ -representation, and thus $\alpha(\theta^\vee) = 0$.
- If $\alpha - \theta \in R$, then $\mathbb{C}X_\alpha \oplus \mathbb{C}X_{\alpha-\theta}$ is an irreducible component of $V(\theta)$ as an \mathfrak{s}_θ -representation, and thus $\alpha(\theta^\vee) = 1$.

This proves Lemma 3. \square

Thus, $\alpha(\theta^\vee)$ is 0 or 1 for all $\alpha \in R_s^+ \Rightarrow \alpha(\theta^\vee)^2 = \alpha(\theta^\vee)$. Therefore, $c = \rho_s(\theta^\vee) = h_s^\vee$ as ρ_s is the sum of all fundamental weights of \mathfrak{g} corresponding to simple short roots and $\theta^\vee = \sum_{i=1}^n a_i^\vee \alpha_i^\vee$. We get, $\nu + c\Lambda_0 = \rho_s + h_s^\vee \Lambda_0 = \hat{\rho}_s$ which proves the Fact 1. \square

Finally we get the proof of (\Leftarrow) in Proposition 10 by replacing ρ, ρ_s, R_s^+ and R_l^+ by $\hat{\rho}, \hat{\rho}_s, \hat{R}_s^+$ and \hat{R}_l^+ respectively in Proof of Case 2 in Proposition 9. \square

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